Mathematical Challenge January 2017

Chaos

References

- [1] Devaney, Robert L. An introduction to chaotic dynamical systems. Vol. 13046. Reading: Addison-Wesley, 1989.
- [2] Strogatz, Steven H. Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering. Westview press, 2014.
- [3] Casdagli, Martin (1991). "Chaos and Deterministic versus Stochastic Non-linear Modelling". Journal of the Royal Statistical Society, Series B. 54 (2): 303–328. JSTOR 2346130
- [4] Lesne, Annick. "Chaos in biology." Rivista di Biologia Biology Forum. Vol. 99. No. 3. ANICIA SRL, 2006.

Description

Chaos is a property of some dynamical systems that is generally loosely defined as "sensitivity to initial conditions". More precisely, considering the *Lyapunov exponents* $\vec{\lambda}(x_0)$ of the dynamic $x(t) = f(t; x_0)$ as the logarithm of the eigenvalues $\vec{\Lambda}(x_0)$ of

$$L(x_0) = \lim_{t\to\infty} (J^t(x_0) \cdot J^{t,+}(x_0))^{1/2t}, \quad J^t(x_0) = \frac{df^t}{dx}(t, x_0),$$

"sensitivity to initial conditions" at the starting point x_0 corresponds to the existence of a strictly positive Lyapunov exponent. This sensitivity makes the long term prediction of chaotic dynamics impossible if the initial condition of the system is not known exactly. Note that as opposite to stochastic systems is not randomness, that makes chaotic system unpredictable. Chaotic systems are actually intrinsically very different from stochastic systems. Indeed, in general, chaotic systems are deterministic and they look stochastic at first sight just due to their dynamic feature and our imperfect knowledge of the initial state. A stricter definition of a chaotic dynamics [1], relevant for our considerations, includes two additional properties (under some conditions, possibly redundant)

- *Topologically mixing* systems: The time evolution of any open set in the phase space overlaps with any other set.
- Systems with *dense periodic orbits*: For every point in the phase space, there is a point arbitrarily close belonging to a periodic orbit.

Chaotic dynamics can arise in pretty simple dynamic systems [2]. A necessary condition for finite dimensional systems to show chaotic behaviour is however a non-linear dynamics. One of the simplest and most known examples is the *logistic map* $x_{n+1} = ax_n(x_n - 1)$. This illustrates as well how a class of parametrized systems, can show different behavior, among which chaotic dynamics, for different parameter values. For $a \in [0,3]$ all paths converge towards a fixed point *attractor* independently from the initial position. For $a \in (3, 3.56995)$ the fixed point bifurcates and the system almost always converges towards a periodic path. Above $a = 3.56995$ the system alternates parameter intervals for which the dynamics shows sensitivity to initial conditions and aperiodicity with intervals (‘*islands of stability*’) where the system converges towards periodic cycles of increasing periodicity. For $a = 4$ the system shows chaotic behaviour with dense periodic cycles of any length.

Note that, while the logistic map with $a = 4$, shows chaotic behavior everywhere on [0,1], some other systems show chaos just on a subset of the phase space. Most notable examples of this behavior is when chaos onsets on attractors. In this case, any initial position on the basin of attraction of this attractor will converge towards a chaotic dynamics. In these circumstances the attractor is often a *strange attractor,* i.e. they has a fractal structure. The Poincaré-Bendixson theorem however requires that the dimension of the dynamical system has to be bigger than 2 for strange attractors to exist.

Two possible approaches to show that a real system is chaotic consist in either assuming a model (possibly with chaotic dynamics) and show that the model succeeds in explaining observed features or performing tests on real observed data [3]. However, due to the data paucity, the intrinsic measurement noise, non-stationary behaviors and lack of controllable experiments, it is often difficult to provide conclusive evidence of chaotic behavior in real systems. While keeping this caveat in mind, which is especially applicable to biological systems, for which, furthermore, the intrinsic dimensionality of the system is far from obvious, we can assert that there are growing indications that that often a healthy functioning of some biological systems is associated to “controlled” chaotic systems, whereas pathological one can correspond to regular rhythms [4]. Examples of this principle are the cardiac rhythm and neurons activities.

Questions:

- Q1. Why could a chaotic system represent a competitive advantage in biological systems in terms of stability and flexibility?
- Q2. Why chaotic dynamics are interesting from an evolutionary perspective?

Biological systems have always been a precious source of inspiration for technological innovations. This is true also for learning and optimization techniques, indeed techniques like genetic algorithms, ant colony optimization, neural networks, ensemble methods,… exploit possibly simplified and idealized versions of well established biological approaches. Therefore, a natural question to ask is if the same advantages that chaotic systems provide to biological mechanisms can possibly be exploited in a technological context

Questions:

- Q3. How can the above advantages of chaotic systems be exploited in computational tasks like optimization and machine learning?

We look forward to your opinions and insights. Sebastiano Rossi is responsible for the distribution of all results to the Methodology Board.

Best Regards,

swissQuant Group Leadership Team