A Simple Model of Credit Contagion *

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ABSTRACT

We propose a simple model of credit contagion in which we include macro- and microstructural interdependencies among the debtors within a credit portfolio. We show that even for diversified portfolios, moderate microstructural interdependencies have a significant impact on the tails of the loss distribution. This impact increases dramatically for less diversified microstructures. Since the inclusion of microstructural interdependencies acts on the tails, the choice of an appropriate risk measure for credit risk management is a delicate task.

JEL Classification Codes: C19, C69, G18, G21.

Key Words: Credit Portfolio Risk Management, Contagion, Macroeconomic and Microstructural Interdependence, Value-at-Risk, Expected Shortfall
I. Introduction

A major concern for measuring and managing risk of credit portfolios is the high degree of defaults’ dependence. Such dependence may be caused by both macro- and microstructural channels. The macrostructure relates to the sensitivity of debtors to common factors, such as, e.g., changes in macroeconomic or industry-sector specific fundamentals. For portfolios that are not well diversified in their common factors, such a macrostructural interdependence causes defaults to be positively correlated. Recent examples are the banking problems in Japan, the Asian crisis, and the Russian meltdown. However, pure macrostructural models can hardly accommodate the high degree of correlation found in empirical data on defaults.

The microstructure captures interdependencies between debtors that go beyond their exposure to common factors, e.g., business or legal interdependencies. These microstructure interdependencies can lead to so-called “contagion.” Hence, in our model we define contagion as a transmission effect that underlies a microstructure interdependence. Contagion risk is then the risk that through microstructural channels, the credit deterioration of a counterparty triggers the credit deterioration of other counterparties.

To account for the two interdependence structures, we embed two different types of data into a model for credit contagion. First, we calibrate our model to statistical data represented by a credit migration or transition matrix. Credit migration matrices characterize the past changes in the credit quality of debtors. These matrices are cardinal inputs to most risk management applications, including portfolio risk assessment, and the calculation of economic and regulatory capital requirements. We obtain transition matrices either by using internal models or from rating agencies such as Moody’s and Standard & Poor’s. The matrix entries measure default as the average frequency with which debtors of the same rating have defaulted. (But we note that there are many other ways to define credit migration.)

As a second input to our model, we use additional counterparty-specific data. These data contain accounting figures, business data, or expert knowledge based on self-assessment by customer advisors. Some of the data are the bank’s private information, but the data are often also publicly available. E.g., large real estate companies in Switzerland explicitly state their
business interdependence on those renters that are responsible for more than five percent of their total income. We model such microstructural information by using weighted graphs.

Historical credit data interweaves macro- and microstructural causes for default and rating migration. It is difficult to assess how much of the empirical default and transition probabilities can be attributed to macroeconomic causes or to firm-level economic distress that grows through microstructural channels. Moreover, to fine-tune the statistically calibrated rating matrix, the rating processes in a financial institution incorporate some additional quantitative or subjective assessments of the debtor’s business environment. Therefore, if a financial institution estimates its own transition matrix from internal historical data, the matrix will inevitably contain both types of causes. Since rating agencies consider large populations of global and mostly medium- or large-sized firms, we claim that their migration matrix is approximatively independent of microstructural effects. Hence, the calibration of our macrostructure model hinges on the use of such a diversified transition matrix, one that is approximatively free of microstructural effects.

In this paper, we consistently incorporate both micro- and macrostructural interdependencies. This integration allows us to fully investigate the impact of microstructural interdependencies on credit portfolios and to identify their marginal contribution to the overall risk of a credit portfolio. We find that microstructural interdependence significantly increases the correlation among debtors and fattens the tails of the credit loss distribution. Even for well-diversified credit portfolios with moderate microstructural interdependencies, we find striking evidence for a significant increase in rating-count volatility and migration speed. We can attribute the high volatility to feedback effects channelled through business interdependencies, even if these are moderate. If we were to use a pure macrostructural approach, such an approach would totally neglect all these effects, and therefore substantially underestimate the credit portfolio’s risk.

Finally, since business interdependencies are mostly tail effects, our findings hint at the danger of using value-at-risk (VaR) to manage credit risk and give strong support for tail-sensitive risk measures such as expected shortfall.
Credit risk modeling has enjoyed a rapid growth during the last decade. There is a vast number of different credit risk models. We do not attempt to give an exhaustive survey on the literature, but refer to the books of Bielecki and Rutkowski (2002) and Duffie and Singleton (2003). However, the literature so far has paid little attention to microstructural interdependence. Only in recent years have we seen a growing interest in more refined interdependence structures for credit risk modeling. These endeavors broadly fall into two categories. The first category aims at pricing multi-name credit derivatives. Recent contributions are, e.g., Jarrow and Yu (2001) and Yu (2005).

The second category is aimed at risk management applications. To our best knowledge, the first publications which address contagion risk are Davis and Lo (1999) and (2001) proposing infectious defaults. Giesecke and Weber (2004) introduce credit contagion models on the basis of interacting particle systems. Furthermore, Giesecke and Weber (2005) make a first attempt to integrate macroeconomic effects, causing cyclical default correlations, and contagion phenomena, associated with the local interaction of debtors with their business partners. However, as they have to restrict the focus of their model to equilibrium distributions, their analysis is essentially static. Another equilibrium approach is Horst (2005), who proposes an equilibrium model of credit ratings, where the risk to large portfolios is intrinsic in that it cannot be diversified away. Focardi and Fabozzi (2004) extend the lattice approach of Giesecke and Weber (2005) to a contagion model using percolation and random graphs. In an idealized framework, they are able to show the impact of the credit portfolio’s internal interdependency structure on the shape of the aggregate loss distribution.

Our approach is based on Markov chain models. Markov chain models for rating migrations date back to the work of Jarrow, Lando and Turnbull (1997), where they are used for pricing credit risky securities. Kijima, Komoribayashi and Suzuki (2002) extended the Jarrow–Landow–Turnbull model for the purpose of risk management. Our work in turn extends the latter one by modeling a microstructural dependence in addition to the usual macroeconomic dependence among the debtors. In a similar vein, Frey and Backhaus (2003) present within a Markovian setting an approach that incorporates interacting defaults and counterparty risk. Their approach is based on intensity-based credit models and uses results from mean-field the-
ory. However, it remains unclear how their model is calibrated to large and heterogeneous portfolios. We follow a more direct approach, without relying on mean-field considerations. We incorporate microstructural data that are available in the bank’s credit risk department. By using a directed graph representation to represent pairwise links between obligors, we can construct a topological risk map of the bank’s credit portfolio and we show how to embed a detailed empirical description of these links. Therefore, from a practical perspective, possible infections within the portfolio can be detected, traced back to a single counterparty, and hence counteracted in the most appropriate way.

Only recently, Carling, Rönnegård and Roszbach (2004) tested a General Linear Mixed Model on two Swedish banks’ business loan portfolios over the period 1996-2000, taking into account dependencies between firms from both common factors and industry specific errors. Their main findings support our modeling approach. First, they find that by including industry specific errors the estimates of individual default risk are little affected. Second, the value-at-risk figures for credit losses however will increase by 50-200 percent. This increase is consistent with our numerical results.

Our paper is structured as follows. Section II presents our approach to embed microstructural interdependencies into a macrostructural model for credit risk. Section III shows how to calibrate our model to macroeconomic data. In Section IV, we explore the microstructure by using numerical examples. Section V concludes.

II. The Model

The building blocks of our model are the transition matrices and the availability of debtor specific data on business interdependencies. In what follows, we unify these two blocks within a consistent framework for credit risk modeling.
A. Rating Dynamics

The creditworthiness of a debtor is given by her rating. We describe the rating dynamics by a stationary discrete-time Markov chain. The stationarity assumption can be removed by adding cyclical trends. Indeed, it is widely held that default rates are negatively correlated with real economic activity over the business cycle.\textsuperscript{1} However, without loss of generality, we prefer to work under the stationarity assumption.

Given a debtor \( i \), we describe her creditworthiness by \( d \) different rating classes \( \mathcal{X} = \{1, \ldots, d\} \) with \( d \) the absorbing default state. By assumption, the rating dynamics of a single debtor \( i \) follow a discrete-time process \( X_i = (X_i(t))_{t \geq 0} \). Given a credit portfolio with \( N \) different debtors, the joint rating dynamics \( X(t) \) follows the Markov chain \( X(t) = (X_1(t), \ldots, X_N(t))' \in \mathcal{X}^N \). The transition probabilities between the different rating classes are summarized in a transition matrix \( T_{xy} \). Formally, we write for a transition from a portfolio state \( x = (x_1, \ldots, x_N) \) to a state \( y = (y_1, \ldots, y_N) \)

\[
T_{xy} = \mathbb{P}[X(t+1) = y \mid X(t) = x]. \tag{1}
\]

So far, we have defined expression (1) on the portfolio level. This generality is not feasible in practice if, say, several thousand debtors are within a portfolio. To reduce the complexity, we follow a standard approach and model the rating migrations of the debtors \( i = 1, \ldots, N \) as conditionally independent of some random vector \( Z \). A conditional independence structure is not only more intuitive than a generic model (1), but also computationally more attractive. Such a conditioning procedure leads to the following product structure for the transition matrices

\[
T_{xy} = \mathbb{E}\left[ \prod_{i=1}^{N} T_{x_i y_i | Z} \right], \tag{2}
\]

where

\[
T_{x_i y_i | Z} = \mathbb{P}[X_i(t) = y_i \mid X_i(t-1) = x_i, Z], \tag{3}
\]
is the individual conditional transition matrix of debtor \( i \). Consequently, equation (2) translates the interdependence structure between the debtors into an interdependence between the coordinates of \( Z \).

We construct the causal model behind the rating dynamics in two steps. In the first step, we model the macroeconomic causality in terms of a latent variable model. In the second step, we further decompose the idiosyncratic part of the latent variable to incorporate the microstructure.

**B. Macrostructure**

Once the conditional independence structure and the conditional migration matrices \( T^{(i)}|Z \) in (2) are determined, we have fully specified the dynamics of the rating process. To simplify (2) further, we group the debtors into a small number \( K \) of different sectors according to some classification scheme. The resulting classification function

\[
s: \{1, \ldots, N\} \to \{1, \ldots, K\},
\]

maps every debtor to one of these sectors and reduces equation (2) to

\[
T_{xy} = \mathbb{E} \left[ \prod_{i=1}^{N} T^{s(i)}_{x_i,y_i}|Z \right].
\]

To model the macrostructure effects, we assume that the independence generating variables \( Z \) are given by systematic sector-specific risk factors

\[
Z = Z = (Z_s)_{s=1,\ldots,K} \sim N(0, \Lambda), \quad \Lambda = (\lambda_{i,j})_{i,j=1,\ldots,K},
\]

and denote by

\[
Y = (Z_{s(i)})_{i=1,\ldots,N} \sim N(0, \Gamma), \quad \Gamma = (\lambda_{s(i),s(j)})_{i,j=1,\ldots,N},
\]

the vector of systematic factors on the debtor specific level and its covariance matrix.
We express the conditional migration matrices $T_{s|Z}$ in terms of a latent variable model, very similar to generalized linear models. For each debtor $i$, we define a synthetic asset return $A_i$ given by a univariate standard Gaussian latent variable

$$A_i = \sqrt{1 - w_{s(i)}^2} Z_{s(i)} + w_{s(i)} \epsilon_i, \quad A_i \sim N(0, 1). \tag{8}$$

The synthetic return in equation (8) consists of a sector-specific and an idiosyncratic debtor-specific component. The vector $\epsilon = (\epsilon_i)_{i=1,\ldots,N} \sim N(0, 1_N)$ collects debtor-specific factors within the credit portfolio. By construction, the $\epsilon_i$ are mutually independent. The debtor-specific factors enter the synthetic asset return with sector specific weights

$$w = (w_s)_{s=1,\ldots,K}. \tag{9}$$

The vector $Z$ represents the synthetic risks of the different sector classes. We assume that the vectors $Z$ and $\epsilon$ are independent. Consequently, $Z$ serves as a conditional independence structure for the latent variable $A_i$.

To model the credit migration matrix, we assume that a credit migration from a rating class $x$ to a new class $y$ is triggered whenever $A_i$ exceeds some threshold value $\theta_{xy}$,

$$-\infty = \theta_{xd+1} \leq \theta_{xd} \leq \ldots \leq \theta_{x2} \leq \theta_{x1} = \infty. \tag{10}$$

Thus, for a debtor $i$, the transition probability of migrating at time $t$ from class $x$ into a new rating class $y$ at time $t+1$ is

$$T_{xy}^{(i)} = \mathbb{P}(X_i(t+1) = y \mid X_i(t) = x) = \mathbb{P}(A_i \in [\theta_{x(y+1)}, \theta_{xy})). \tag{11}$$

The threshold values $\theta = (\theta_{xy})$, the correlation matrix $\Lambda$ and the specific risk weights $w_s = (w_1, \ldots, w_K)$ are free model parameters, which we calibrate to historical default data.
C. Integrating Microstructure

So far, the return specification in equation (8) only allows us to incorporate macrostructural interdependence or sector-specific risks. However, we wish to explicitly model business interdependencies. Therefore, we provide a different specification of the return $A_i$. To do so, we introduce some further notation.

Given $N$ debtors, we write the macrostructural (or sector-specific) weights on the individual debtor level as

$$v = (w_{s(i)})_{i=1,...,N}. \quad (12)$$

Also on the debtor level, we indicate the strength of the business interdependence between two counterparties with the matrix

$$\Xi = (\xi_{ij})_{i,j=1,...,N}, \quad (13)$$

which we call the “business matrix.” As a proxy for the weights $\xi_{ij}$, we could use the business volume of debtor $j$ with one of its non-substitutable counterparties $i$. We restrict our analysis to positive business relations. For all debtors without a significant microstructural interdependence, we assume that the weights are set to zero. Furthermore, since a firm does not self-affect itself, we assume $\xi_{jj} = 0$ for $j = 1, \ldots, N$. The vector

$$\eta = (\eta_i)_{i=1,...,N} \quad (14)$$

finally attaches weights to the residual risk of each debtor. The quantity $\eta_i$ is the idiosyncratic or obligor specific risk weight, which is not assigned to microstructure effects. In contrast to the macrostructural setup in Section [B] where the parameters are statistically calibrated, the microstructural model parameters $\Xi$ and $\eta$ are determined by expert knowledge, reflecting the bank’s credit expertise.3

With the notation at hand, we formalize the microstructure for a credit portfolio with $N$ debtors.
Definition 1. A microstructure for a collection of counterparties \( C = \{1, \ldots, N\} \) is a directed weighted graph \( G = (C, E, \Xi, \eta) \). The nodes correspond to the counterparties. A directed weighted edge \( E_{ji} \)

\[
\eta_j \xrightarrow{\xi_{ij}} \eta_i
\]

indicates a business relation from \( j \) to \( i \) that induces a counterparty risk for firm \( i \) with strength \( \xi_{ij} \) given by the edge weights. The node weight \( \eta_i \) represents the residual risk of debtor \( i \).

Figure 1 shows where we are headed. Panel (A) illustrates the interdependencies in a pure macrostructural model. The credit rating depends only on the macrostructural variables \( Z_s(i) \) and \( Z_s(j) \). Firm \( i \) cannot be directly influenced by the variable \( Z_s(j) \), nor can firm \( j \) be influenced by \( Z_s(i) \). The only interdependence is generated through \( \Gamma \). However, by introducing a business interdependence \( \xi_{ij} \), the credit rating of firm \( i \) now becomes dependent on both the macrostructural variable \( Z_s(j) \) and the idiosyncratic risk \( \epsilon_j \) through her business relation with firm \( j \). At the same time, the residual firm-specific idiosyncratic risk for firm \( i \) reduces to \( \eta_i \).

We note that our model also allows asymmetric effects. Since we work in a directed graph, we can impose a weight \( \xi_{ji} \) which differs from \( \xi_{ij} \). Then, the financial health of firm \( j \) can be infected by \( Z_s(i) \) through her business relation with firm \( i \).

In general, the collection of counterparties includes all firms or individuals which affect the creditworthiness of debtors. Hence, firms that are not the bank’s debtors also contribute indirectly to the bank’s credit risk through business relations with other debtors.

To integrate a microstructure \( G = (C, E, \Xi, \eta) \) into a macrostructural model, we set \( A = (A_i)_{i \in C} \) for the vector of all synthetic asset returns. Furthermore, if \( v \) is a vector, \( D(v) \) is the diagonal matrix with diagonal elements \( v \). If \( f \) is a real valued function of a real variable then \( f(D(v)) = D(f(v)) \), where \( f(v) = (f(v_i)) \) is defined componentwise. If \( \Gamma \) is a square matrix, then \( D(\Gamma) \) denotes the diagonal matrix with the same diagonal as \( \Gamma \). With these conventions, we can write the macrostructural model in equation (8) in a more compact form as

\[
A = D\left(\sqrt{1 - v^2}\right) Y + D(v) \epsilon. \tag{15}
\]
To integrate the macro- and microstructure, we assume that a financial institution is able to correctly assess the unconditional rating migration probabilities of individual debtors. Therefore, at the portfolio level, if we add macro- and microstructural interdependencies between debtors, we need to gauge our model accordingly, such that the unconditional default probabilities for individual debtors are left unchanged. Only then, adding macro- and microstructure interdependency at the portfolio level does not change the migration probabilities at the individual debtor level. However, the new interdependencies may alter the conditional default probabilities.

Therefore, when we introduce microstructural dependency by replacing $\epsilon$ in equation (15) by a normalized function $\varepsilon(\mathcal{G}, Y, \epsilon)$, we need to impose the following compatibility conditions:

(C1) Compatibility with macrostructural specific risk. A sufficient condition is that $D_\varepsilon \varepsilon(\mathcal{G}, Y, \epsilon)$ is normalized to standard normal.

(C2) Compatibility with exogenously prescribed transition probabilities. A sufficient condition is that the marginals of $A$ remain standard normal.

Hence, to consistently add microstructural interdependence, we replace equation (15) by

$$A = D_Y D \left( \sqrt{1 - \eta^2} \right) Y + D_\varepsilon D (\eta) \varepsilon(\mathcal{G}, Y, \epsilon),$$

where the two normalizing diagonal matrices, $D_Y$ and $D_\varepsilon$ must be determined such that the conditions (C1) and (C2) are fulfilled. We note that the pure macrostructural specification in (15) is a special case of the specification in (16) with $\varepsilon(\mathcal{G}, Y, \epsilon) = \epsilon$ and $D_Y = D_\varepsilon = 1$.

C.1. Recursive Integration of Microstructure

There is no canonical way to specify the microstructural dependence in the idiosyncratic term $\varepsilon(\mathcal{G}, Y, \epsilon)$. For our subsequent numerical analysis, we specify $A$ as having a direct linear effect on $\varepsilon(\mathcal{G}, Y, \epsilon)$, i.e., we set

$$\varepsilon(\mathcal{G}, Y, \epsilon) = \Xi A + D (\eta) \epsilon.$$

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Then,

\[ A = D_Y D \left( \sqrt{1 - v^2} \right) Y + D_\varepsilon D (v) (\Xi A + D (\eta) \epsilon), \] (18)

where \( D_Y \) and \( D_\varepsilon \) must be determined such that conditions (C1) and (C2) are fulfilled.

Representation (18), although simplified, already provides a rich structure for business interdependencies. It allows us to take into account both indirect and cyclic influences. We remark that even the linear specification in (17) will result in highly nonlinear effects for the risk calculations. A possible situation is stylized below.

The asset return of firm \( i \) might affect firm \( j \)'s asset return. The business interdependence strength \( \xi_{ji} \) indicates the infection intensity. Note that the effects do not need to be symmetric, i.e., a negative shock for firm \( j \) can have a positive effect for firm \( i \). Such asymmetric effects can be incorporated by adjusting the interdependence \( \xi_{ji} \) accordingly.

Firms \( j \) and \( i \) can be members of different sectors, and so differ in their dependence on macroeconomic variables. Therefore, firm \( j \) might import a microstructural shock from a different sector via the microstructure channel \( E_{ij} \). In addition, feedback effects could occur. As an example, we consider an idiosyncratic event that hits firm \( i \) and deteriorates her financial health and possibly her rating. Then, this deterioration affects firm \( j \) through her business relation with \( i \). Since firm \( j \) has business relations with firm \( k \), firm \( k \)'s financial health also deteriorates. If firm \( k \) has business relations with firm \( i \), firm \( i \)'s financial health deteriorates even further. Hence, the idiosyncratic event for firm \( i \) induces a feedback.

We rewrite (18) as

\[ A = C_Y Y + C_\varepsilon \epsilon, \] (19)


\[
C_Y = \left(1 - D_\varepsilon D (v) \Xi \right)^{-1} D_Y D \left( \sqrt{1 - v^2} \right), \quad C_\varepsilon = \left(1 - D_\varepsilon D (v) \Xi \right)^{-1} D_\varepsilon D (v \eta).
\]

We note that the matrix \((1 - D_\varepsilon D (v) \Xi)^{-1}\) is invertible, because the operator norm of \(D_\varepsilon D (v) \Xi\) is strictly less than one. (All matrix elements are positive and the row sums are strictly smaller than one.) Equation (19) implies that \(A\) is still a centered Gaussian vector with covariance

\[
\text{Cov}(A, A) = C_Y \Gamma C_Y^\top + C_\varepsilon C_\varepsilon^\top. \quad (20)
\]

To guarantee consistency with (C1) and (C2), we proceed in two steps. In the first step, we choose \(D_\varepsilon\) such that the idiosyncratic term \(D_\varepsilon \varepsilon (G, Y, \varepsilon)\) in (18) has unit variance. To this end, we calculate the covariance of \(\varepsilon (G, Y, \varepsilon)\), which we denote by \(\Sigma\), as

\[
\Sigma = \Xi \left( C_Y \Gamma C_Y^\top + C_\varepsilon C_\varepsilon^\top \right) \Xi^\top + D (\eta^2) + \Xi C_\varepsilon D (\eta) + D (\eta) C_\varepsilon^\top \Xi^\top. \quad (21)
\]

In the second step, we choose \(D_Y\) such that \(A\) in (19) has unit variance. Therefore, for the recursive integration of microstructural interdependencies, the conditions (C1) and (C2) read

\[
\left\{
\begin{array}{l}
(C1) : \quad D_\varepsilon^2 D (\Sigma) = 1, \\
(C2) : \quad D (C_Y \Gamma C_Y^\top + C_\varepsilon C_\varepsilon^\top) = 1.
\end{array}
\right. \quad (22)
\]

We note that these two normalization steps are dependent, since they result in coupled equations for both \(D_Y\) and \(D_\varepsilon\). The coupling follows from the dependence of \(\Sigma\) on both \(C_Y\) and \(C_\varepsilon\) and, hence, on \(D_Y\) and \(D_\varepsilon\) (see equation 21). So far, we are not aware of a closed-form solution to (22). For large credit portfolios, finding a numerical solution for equation (22) might turn out to be tedious. Furthermore, since the matrix \(\Xi\) is sparse, the required inversion \((1 - D_\varepsilon D (v) \Xi)^{-1}\) can be numerically unstable.
C.2. Approximated Recursive Integration of Microstructure

We can circumvent the inversion of \((1 - D_\varepsilon D(v) \Xi)^{-1}\) by using an approximative procedure. Instead of recursively integrating the microstructure as in equation (18), we replace \(A\) on the right hand side of equation (18) by the pure macroeconomic effects. Writing the macroeconomic synthetic return as

\[
A^{(0)} = D \left( \sqrt{1 - v^2} \right) Y + D(v) \epsilon, \tag{23}
\]

we define the idiosyncratic part as

\[
\varepsilon^{(1)}(G, Y, \epsilon) = \Xi A^{(0)} + D(\eta) \epsilon \tag{24}
\]
\[
= \Xi D \left( \sqrt{1 - v^2} \right) Y + (\Xi D(v) + D(\eta)) \epsilon \tag{25}
\]
\[
\equiv E^{(1)}_Y Y + E^{(1)}_\epsilon \epsilon. \tag{26}
\]

Calculating the covariance matrix of \(\varepsilon^{(1)}(G, Y, \epsilon)\),

\[
\Sigma^{(1)} = E^{(1)}_Y Y E^{(1)}_Y + E^{(1)}_\epsilon E^{(1)}_\epsilon, \tag{27}
\]

we observe that equation (27) is independent of the normalization factors \(D^{(1)}_Y\) and \(D^{(1)}_\varepsilon\). This property allows us to explicitly derive \(D^{(1)}_\varepsilon\) such that we are consistent with condition (C1).

In the second step, we choose \(D^{(1)}_Y\) such that \(A^{(1)}\) has unit variance. Given the specification of the idiosyncratic term, we can write the first-order approximation for the synthetic asset return as

\[
A^{(1)} = D^{(1)}_Y D \left( \sqrt{1 - v^2} \right) Y + D^{(1)}_\varepsilon D(v) \varepsilon^{(1)}(G, Y, \epsilon). \tag{28}
\]

Collecting terms, we can express \(A^{(1)}\) as

\[
A^{(1)} = C^{(1)}_Y Y + C^{(1)}_\epsilon \epsilon, \tag{29}
\]
where
\[
C_Y^{(1)} = D_Y^{(1)} D \left( \sqrt{1 - v^2} \right) + D_\varepsilon^{(1)} D (v) E_Y^{(1)}, \tag{30}
\]
\[
C_\varepsilon^{(1)} = D_\varepsilon^{(1)} D (v) E_\varepsilon^{(1)}. \tag{31}
\]

We calculate the covariance of $A^{(1)}$ as
\[
\text{Cov} \left( A^{(1)}, A^{(1)} \right) = \left( D_Y^{(1)} D \left( \sqrt{1 - v^2} \right) + B^{(1)} \right) \Gamma \left( D_Y^{(1)} D \left( \sqrt{1 - v^2} \right) + B^{(1)\top} \right) + C_\varepsilon^{(1)} C_\varepsilon^{(1)\top}, \tag{32}
\]
where, for notational convenience, we define $B^{(1)} = D_\varepsilon^{(1)} D (v) E_Y^{(1)}$. We note that since $\Gamma$ is symmetric, we have $D (B^{(1)\Gamma}) = D (\Gamma B^{(1)\top})$. Therefore, condition (C2), which requires $\text{Var} \left( A^{(1)} \right) = D \left( \text{Cov} \left( A^{(1)}, A^{(1)} \right) \right) = 1$, implies that $D_Y^{(1)}$ is the solution to the quadratic matrix equation
\[
1 = (D_Y^{(1)})^2 D \Gamma \left( 1 - v^2 \right) + 2 D_Y^{(1)} D \left( \sqrt{1 - v^2} \right) D \left( \sqrt{1 - v^2} \right) + D \left( C_\varepsilon^{(1)} C_\varepsilon^{(1)\top} + B^{(1)\Gamma B^{(1)\top}} \right)
\equiv Q(D_Y^{(1)}, D_\varepsilon^{(1)}). \tag{33}
\]

Equation (34) can explicitly be solved for $D_Y^{(1)}$.

Summarizing the above analysis for the first-order approximated recursive integration model, the normalizing matrices that guarantee consistency with the conditions (C1) and (C2) read
\[
\begin{cases}
(C1) & (D_\varepsilon^{(1)})^2 = D \left( \Sigma^{(1)} \right)^{-1}, \\
(C2) & 1 = Q \left( D_Y^{(1)}, D_\varepsilon^{(1)} \right). \tag{35}
\end{cases}
\]

Equation (28) is a first-order approximation to equation (18) that allows us to express the asset return by an explicit equation. However, the first-order model does not account for any reverse or back-propagation effects. To capture possible feedback effects, we must consider at least second-order approximations. We obtain the $n$-th order approximations $A^{(n)}$ by recursively plugging $A^{(n-1)}$ into the definition of the idiosyncratic part $\varepsilon^{(n)}(G, Y, \epsilon)$. 

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Proposition 2. Consider the \( n \)-th order approximation of the return \( A \),

\[
A^{(n)} = D^{(n)}_Y D \left( \sqrt{1 - v^2} \right) Y + D^{(n)}_\varepsilon D(v) \varepsilon^{(n)}(\mathcal{G}, Y, \varepsilon),
\]

where \( \varepsilon^{(0)}(\mathcal{G}, Y, \varepsilon) = \varepsilon \) and \( D^{(0)}_Y = D^{(0)}_\varepsilon = 1 \). The idiosyncratic part is given as

\[
\varepsilon^{(n)}(\mathcal{G}, Y, \varepsilon) = E^{(n)}_Y Y + E^{(n)}_\varepsilon \varepsilon,
\]

with

\[
E^{(n+1)}_Y = \sum_{i=0}^{n} \left( \prod_{j=0}^{n-i-1} \Xi D^{(n-j)}_\varepsilon D(v) \right) \Xi D^{(i)}_Y D \left( \sqrt{1 - v^2} \right),
\]

\[
E^{(n+1)}_\varepsilon = \sum_{i=0}^{n} \left( \prod_{j=0}^{n-i-1} \Xi D^{(n-j)}_\varepsilon D(v) \right) D(\eta) + \prod_{i=0}^{n} \Xi D^{(n-i)}_\varepsilon D(v).
\]

The normalizing matrices \( D^{(n)}_Y \) and \( D^{(n)}_\varepsilon \) solve

\[
\begin{cases}
(C1) & (D^{(n)}_\varepsilon)^2 = \mathbf{D} \left( \Sigma^{(n)} \right)^{-1}, \\
(C2) & 1 = Q \left( D^{(n)}_Y, D^{(n)}_\varepsilon \right).
\end{cases}
\]

We present the proof of Proposition (2) in the Appendix. Finally, we note that the approximated recursive integration model presented in this section converges to the recursive integration model of Section C.1.

C.3. A Two-Firms Example

To clarify the impact of the microstructure on the returns \( A_i \) and the covariance matrix, we briefly consider a stylized model of two firms. We assume that only firm 1 has a business relation with firm 2, but not vice versa. Therefore, we have \( \xi_{12} \neq 0 \), whereas all other entries in the business matrix are zero. Hence, \( \eta_1 = 1 - \xi_{12} \) and \( \eta_2 = 1 \). Furthermore, we assume that the
two companies are exposed to different sector risks $Z_1$ and $Z_2$. Then, for the macroeconomic synthetic return, we can simply write

$$
A_1^{(0)} = \sqrt{1 - v_1^2} Z_1 + v_1 \epsilon_1,
A_2^{(0)} = \sqrt{1 - v_2^2} Z_2 + v_2 \epsilon_2.
$$

Using the results from Section C.2, we can write the first-order approximation of the returns as

$$
A_1^{(1)} = \omega_{Z_1} \sqrt{1 - v_1^2} Z_1 + \omega_{Z_2} Z_2 + \frac{v_1 (1 - \xi_{12})}{\sqrt{1 - 2(1 - \xi_{12}) \xi_{12}}} \epsilon_1 + \frac{v_1 v_2 \xi_{12}}{\sqrt{1 - 2(1 - \xi_{12}) \xi_{12}}} \epsilon_2,
A_2^{(1)} = \sqrt{1 - v_2^2} Z_2 + v_2 \epsilon_2,
$$

where

$$
\omega_{Z_1} = \frac{v_1 \xi_{12} \lambda_{12} + \sqrt{1 - 2(1 - \xi_{12}) \xi_{12} - v_1^2 (1 - v_2^2) \xi_{12}^2 \lambda_{12}^2}}{\sqrt{1 - 2(1 - \xi_{12}) \xi_{12}}},
\omega_{Z_2} = \frac{v_1 \sqrt{1 - v_2^2} \xi_{12}}{\sqrt{1 - 2(1 - \xi_{12}) \xi_{12}}}.
$$

Note that the macroeconomic dependency only shows up in the expression for $\omega_{Z_1}$ that normalizes the influence of $Z_1$ on the return $A_1^{(1)}$. If $\lambda_{12} = 0$, then we would get $\omega_{Z_1} = 1$, whereas the other weights remain unchanged. To isolate the first-order effect on the correlation that is due only to business dependency, we calculate the correlation between $A_1^{(0)}$ and $A_2^{(0)}$, and we set $\lambda_{12} = 0$. We obtain

$$
\text{Cor} \left( A_1^{(1)}, A_2^{(1)} \right) = \frac{v_1 \xi_{21}}{\sqrt{1 - 2(1 - \xi_{12}) \xi_{12}}}. \tag{41}
$$

Obviously, this correlation increases when the (positive) strength of the business relation also increases. The increase depends on the magnitude of $v_1$, since we let the microstructural channels impact the synthetic return through the idiosyncratic part.
III. Calibration to Macroeconomic Data

In this section we calibrate the macrostructural model to historical default rates for the different industry sectors. We call this a “top-down” approach, since we calibrate the individual transition probabilities of every debtor $i$ to a given migration matrix ($T_{xy}$). For the macrostructural model, we calibrate three sets of parameters, i.e., the thresholds ($\theta_{xy}$), the risk weights ($w_k$), and the correlation matrix $\Lambda$.

To fix the threshold values ($\theta_{xy}$) in equation (10), we require the resulting transition probabilities ($\hat{T}_{xy}$) to match the empirical matrix ($T_{xy}$). The standard Gaussian assumption on $A_i$ and equation (11) imply the calibration condition

$$\hat{T}_{xy} = \Phi (\theta_{xy}) - \Phi (\theta_{x(y+1)}) ,$$

with $\Phi(\cdot)$ the cumulative distribution function of the standard normal. Therefore, we fix the default threshold value

$$\theta_{xd} = \Phi^{-1} (\hat{T}_{xd}) ,$$

and calculate the remaining thresholds as

$$\theta_{xy} = \Phi^{-1} \left( \hat{T}_{xy} + \Phi (\theta_{x(y+1)}) \right) .$$

To identify the sector-specific risk weights $w = (w_1, \ldots, w_K)$ for the $K$ sectors, we use a mean value consideration. Because the risk weights do not depend on the rating, we take into account only sector-specific effects and leave aside the debtors’ rating. For a representative debtor in each sector class $s$, we consider the synthetic asset return

$$A_s = \sqrt{1 - w_s^2} Z_s + w_s \epsilon_s ,$$

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where $A_s, \epsilon_s \sim N(0, 1)$, $Z_s \sim N(0, \lambda)$, and $Z_s$ and $\epsilon_s$ are independent. A representative debtor in class $s$ defaults if the asset return falls below a sector-specific threshold value $\vartheta_s$. We define the threshold value $\vartheta_s$ by the unique solution of

$$E[\chi_s] = \Phi (\vartheta_s) = \hat{m}_s.$$  

(46)

where $\chi_s = I_{\{A_s \leq \vartheta_s\}}$ is the default indicator and $\hat{m}_s$ is the mean value of the historical default rate of the class $s$.

To calculate the sector-specific weights $w_s$, we fit the variance $E[\chi_s | Z_s]$ of the conditional default indicator for class $s$ to the empirical volatility $\hat{\sigma}_s^2$ of the historical default rates. In other words, conditioning w.r.t. $Z$ means averaging over the microstructure and idiosyncratic effects. The conditional expectation $E[\chi_s | Z_s]$ is then the corresponding orthogonal projection on the macro variables. Hence, we calculate the variance as

$$\text{Var} (E[\chi_s | Z_s]) = \hat{\sigma}_s^2.$$  

(48)

Since the distribution of $Z_s$ is explicitly known, a nonlinear condition follows

$$\text{Var} (E[\chi_s | Z_s]) = \hat{\sigma}_s^2.$$  

(47)

Using a Newton scheme, we can readily find the solution of the nonlinear equation

$$w_s = f (\hat{m}_s, \hat{\sigma}_s),$$  

(49)

where $w_s$ is a function of the empirical parameters $\hat{m}_s, \hat{\sigma}_s$.

Finally, given the risk weights $w_s$ and the thresholds $\vartheta_s$, we can calibrate the correlation matrix $\Lambda$. To do so, we consider two representative debtors in two sector classes $s_1$ and $s_2$. 

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Because of conditional independence, the mixed moments of the conditional default indicators is then,

\[ \mathbb{E}[\mathbb{E}[\chi_{s_1}|Z]\mathbb{E}[\chi_{s_2}|Z]] = \mathbb{E}[\mathbb{P}(A_{s_1} \leq \vartheta_{s_1}|Z)\mathbb{P}(A_{s_2} \leq \vartheta_{s_2}|Z)] = \mathbb{P}(A_{s_1} \leq \vartheta_{s_1} \land A_{s_2} \leq \vartheta_{s_2}) . \]  \tag{50} 

Because \((A_1, A_2) \sim N(0, R)\) is a bivariate Gaussian with covariance

\[ R = \begin{pmatrix} 1 & \varrho_{s_{12}} \\ \varrho_{s_{12}} & 1 \end{pmatrix} , \] \tag{51} 

we get a nonlinear relation

\[ \Phi_2(\Phi^{-1}(\hat{m}_{s_1}), \Phi^{-1}(\hat{m}_{s_2}), \varrho_{s_{12}}) = \text{Cor}(\mathbb{E}[\chi_{s_1}|Z], \mathbb{E}[\chi_{s_2}|Z]) \sigma(\mathbb{E}[\chi_{s_1}|Z])\sigma(\mathbb{E}[\chi_{s_2}|Z]) + \mathbb{E}[\chi_{s_1}|Z_{s_1}]\mathbb{E}[\chi_{s_2}|Z_{s_2}] . \] \tag{52} 

where \(\Phi\) is the normal distribution function and \(\Phi_2\) is the bivariate normal distribution function. The right-hand side of equation (52) can be entirely expressed in empirical variables, i.e.,

\[ \text{Cor}(\mathbb{E}[\chi_{s_1}|Z], \mathbb{E}[\chi_{s_2}|Z]) \sigma(\mathbb{E}[\chi_{s_1}|Z])\sigma(\mathbb{E}[\chi_{s_2}|Z]) + \mathbb{E}[\chi_{s_1}|Z_{s_1}]\mathbb{E}[\chi_{s_2}|Z_{s_2}] = \hat{\varrho}_{s_{12}}\hat{\sigma}_{s_{1}}\hat{\sigma}_{s_{2}} + \hat{m}_{s_1}\hat{m}_{s_2} . \] \tag{53} 

Again, using Newton’s scheme, we can find a solution for \(\varrho_{s_{12}}\). Performing the same procedure for all other sectors, we end up with the calibrated matrix \(\Lambda\). Once we calculate \(\varrho_{s_{12}} = \text{Cor}(A_1, A_2)\), the correlation \(\lambda_{s_{12}} = \text{Cor}(Z_{s_1}, Z_{s_2})\) is determined by

\[ \lambda_{s_{12}} = \text{Cor}(Z_{s_1}, Z_{s_2}) = \frac{\varrho_{s_{12}}}{\sqrt{1 - \hat{w}_{s_1}^2}\sqrt{1 - \hat{w}_{s_2}^2}} . \] \tag{54} 

IV. Application

We apply our modeling approach presented in Section B and C.2 to different credit-risk portfolios and explore the resulting risk figures.
A. Data and Test Portfolios

To model the macroeconomic structure, we adopt the industry sector classification from the Swiss agency BAK, which uses $K = 14$ sectors (see Table 1). The historical default rates for the 14 industry sectors follow from a time series starting in 1980 and ending in 1997. We adapt the migration matrix from Moody’s migration matrix based on historical default data between 1970 and 2002 (see Table 2). Given the default rates and the migration matrix, we obtain the thresholds ($\theta_{xy}$), the risk weights ($w_k$), and the correlation matrix $\Lambda$ using the calibration procedure outlined in Section III. (See Tables 1, 4, and 5.)

We base our test portfolios on a collection of counterparties grouped at random into the BAK-industry sectors (see Table 1). Each portfolio consists of 102 counterparties. We consider eight rating classes. Rating class 1 corresponds to the highest rating and class 8 to the defaulted class.

We assume that every counterparty has an initial rating different from the default state. We analyze the credit portfolio under four different initial credit qualities, which range from high to average, low, and very low quality. We take the portfolio compositions corresponding to these different qualities from Gordy (2000) (see Table 3).

Now we focus on possible specifications on the portfolios’ microstructure by considering three different types of microstructural interdependencies. The first two types represent extreme cases, whereas the third type of microstructure is taken from an actual credit portfolio of a Swiss bank. The macroeconomic model serves as benchmark.

The first microstructure represents a highly ordered hierarchical group of counterparties in a bank’s portfolio (see Figure 2). We assume two large firms in the portfolio with two distinct groups of direct suppliers or direct service providers. Each of the suppliers has a business volume of 30% with one of the two large firms $A$ and $B$. $A$ and $B$ interact moderately with 10% of their business volume. The direct suppliers of the large firms are nonsubstitutable counterparties for a second set of firms. Again, we assume the business interaction is 30% of total business volume.
We base the second microstructure on a random interdependence pattern between the 102 counterparties. The microstructure is generated by using a random graph (for random graphs, see the seminal work of Erdös and Rényi (1960)). We link the different counterparties in an unstructured way, as shown in Figure 3. However, to guarantee comparability of the random graph with the hierarchical credit portfolio, we randomly select edges with probability $p = 102/(51 \times 101)$ to get an average of 102 edges. We call this portfolio structure a “Diversified Debtor’s Portfolio” (DDP). We refer to the first microstructure as the “Heavy Gravity Portfolio” (HGP). The DDP and HGP are stylized portfolios representing two specific microstructural topologies.

As a third example, we consider what we call the “Real Business Portfolio” (RBP). We structure this portfolio to capture some of the typical structures in credit portfolios. The RBP has a less stylized structure than the HGP or the DDP on the global level, although some subportfolios have a structure similar to a DDP or HGP.

We divide the RBP into five subportfolios: real estate, energy, banking, retail, and weakly-dependent random debtors. The real estate subportfolio is characterized by a dependence structure similar to the HGP. In our example, two large real estate firms build the two central nodes that connect with a large number of renters. Different renters are often independent from each other. However, renters have tenancies with different real estate firms.

Energy companies typically possess a two-level hierarchical interdependence. On the top level, we have energy holdings. Each holding has several investments in power-supply firms. These firms themselves possess investments in power plants. Similar to the real estate case, different holdings may be invested in the same firms and different firms may be invested in the same power plants.

The banking subportfolio is similar to the energy graph, i.e., it may have two or even more levels of interdependence. However, the interdependence no longer obeys a hierarchical order. Rather, the interdependence is horizontal between the different subsidiaries and investments of different banking firms due to large and frequent interbanking activities.
The retail graphs are similar to real estate graphs, but the number of suppliers of the large retail companies is much larger than the number of renters in the real estate graph. Also, the business interdependence goes in the opposite direction. Retail suppliers depend heavily on large companies demanding their goods and services. In contrast, real estate firms depend on their renters. These different dependence directions reflect different market and bargaining power, and eventually affect the credit risk of the bank’s credit portfolio.

The random class gathers interdependencies that are difficult to characterize. Basically, it has the same properties as the DDP, but debtors share only small or negligible business interdependencies.

In addition to the interdependencies outlined above, there are also interdependencies between firms of different subportfolios. For example, a bank that is a renter in a real estate group, has several subsidiaries in the banking sector, and is invested in retail and energy companies. To sum up, the RBP is characterized by several different graph classes and interdependencies between them.

To make the RBP comparable to the DDP and HGP, we assume the same number of nodes and edges. Furthermore, to achieve comparability, we construct the RBP such that the total of the weights in the business matrix equals the total of the DDP and HGP.

B. Simulation Results

We use Monte Carlo simulation to analyze the effect of different interdependence structures on the rating distributions for the portfolios of different credit qualities ranging from “high” to “very low.” Our time horizon is five years with a yearly time interval. For each specification and each time step, we use a sample size of $10^6$ simulations.

B.1. Rating-Count Volatility

By construction, the unconditional default and transition probabilities at the individual counterparty level remain unchanged when we add microstructural interdependencies. Consequently,
the mean number of debtors in different rating states is the same for all four interdependence structures. However, the impact of possible interdependence structures acts on the correlation structure of credit migration.

Figures 4 through 6 plot the additional correlation structure between the synthetic returns $A$ that result from microstructural interdependencies. We express the additional correlation as the absolute difference between the correlations in the pure macrostructural model and the recursive integration model. For each microstructure, we plot the first-order approximation in Panel A. In Panel B, we plot the additional correlation in the full recursive integration model. We find that there is still a substantial amount of correlation generated by higher-order effects. Therefore, for the calculation of risk figures, the first-order approximation, which neglects feedback and looping effects, may only give a very crude approximation of the true underlying risk.

A striking feature of the different figures is how the graphical structure of the debt portfolio adds characteristic patterns to the correlation matrix. The DDP adds correlation in a uniform manner, but hierarchical structures such as the HGP give rise to blockwise increases in correlation. For both the HGP and the DDP, the additional correlation is substantial (up to 0.5). In the RBP, both the DDP and the HGP structures are present in subportfolios. Due to strong interdependencies between debtors with a large number of edges, the increase in correlation reaches 0.8 for some debtors. Furthermore, as we see from Panels A and B of Figure 6, neglecting feedback effects in the RBP heavily underestimates the additional correlation.

Contrary to the mean, we expect the volatility about that mean to be different and dependent on the underlying microstructure. This change is caused by the change in the correlation structure. A higher correlation leads to an increase in the speed of migration through the different categories of the transition matrix. The higher migration speed will eventually have a significant impact on the volatility, and hence on the credit risk calculations.

Figures 7 and 8 compare the volatility surfaces implied by a pure macroeconomic model and different microstructural interdependence models. We parameterize the surfaces by the time horizon and the rating classes. Panels A through D of Figure 7 show the rating volatilities for
an initial portfolio with a high and a very low rating quality, respectively. The lower surface represents the volatility surface in the macroeconomic model and the upper surface represents the volatility in the HGP.

For a portfolio with high quality, we see that microeconomic interdependencies impact the medium-rating classes the most, and that the impact becomes more pronounced with increasing time-horizon. When we consider a portfolio with a very low rating quality, the strongest impact is on the lowest rating class, i.e., on the defaulted firms. Hence, if we exclude microeconomic interdependencies, we would underestimate the volatility of the default class.

Panels A and B of Figure 7 show the rating-count volatility for the DDP. The rating-count volatility is a measure of how many times the debtors within a given portfolio move from one rating class to another one. Panel A assumes a high, and Panel B a very low, initial rating quality. We see that the impacts of microstructural interdependencies exhibit the same patterns for the different portfolio qualities as in the HGP. However, the diversified structure of the business interdependencies in the DDP dampens the impacts of adding microstructural interdependencies. This finding, at least partially, justifies our calibration approach in Section III.

Figure 8 shows the volatility surfaces for the RBP. As we expect from the discussion of the additional correlation in the RBP (see Figure 6), the increase in migration volatility is substantial. Again, the figures show the same qualitative structure as the volatility surfaces in the DDP and the HGP. For high initial portfolio qualities, the inclusion of microstructural interdependencies is most pronounced for medium ratings. For very low initial portfolio qualities, the increase in volatility shows most prominently for the default class. Indeed, the increase almost reaches a factor of two. Such an increase has a significant impact on the calculations of the loss distribution.

B.2. VaR and Expected Shortfall

We examine the effects of our previous findings on the rating-count volatility on the default risk and the loss distribution. We use VaR and expected shortfall as risk measures. We
focus on the probability distribution of potential credit losses. Therefore, we only consider the
ing rating counts in rating class 8 for the macroeconomic model and the different microstructural
interdependencies. To calculate the risk figures, we use four different confidence levels, 95%,
97.5%, 99%, and the 99.5%.

Tables 6, 7, and 8 present the simulation results. All entries are percentage increases
in the corresponding risk figures due to microstructural interdependence. So, e.g., for VaR
we calculate \((\text{VaR}^\text{macro} + \text{micro} - \text{VaR}^\text{macro})/\text{VaR}^\text{macro}\) in percentage numbers and the same for
expected shortfall.

In Table 6, the entries are the risk figures for the DDP. By construction, this portfolio
exhibits a diversified microstructural interdependence with a small impact on the rating-count
volatility. However, a small change in the rating-count volatility already has a major effect on
the resulting risk figures. Depending on the initial portfolio quality and the confidence level,
the expected shortfall of the debt portfolio can be underestimated by as much as 59% compared
to a pure macrostructural model. Using VaR, this difference is slightly less accentuated.

In Table 6 using a diversified microstructure, we find some evidence pointing to the danger
of using VaR as the relevant risk figure. Given a high initial portfolio quality and a one year
horizon, the VaR figure at the 95% confidence level does not pick up any additional risk from
microstructural interdependence. This finding indicates that microstructural interdependence
contributes significantly to the tails of the loss distribution. For the less diversified HGP, these
results become more pronounced (see Table 7). Furthermore, VaR again fails to account for
microstructural interdependence on low quantile levels.

Surprisingly, the effects discussed above for synthetically constructed portfolios are even
stronger in the RBP. We attribute this finding to the fact that the different subportfolios
within the RBP are weakly linked to each other by a small number of large firms. Table 8
shows that for our real-life example, the macroeconomic model underestimates portfolio risk by
more than 193% of expected shortfall for the one-year horizon and a confidence level of 99.5%.
Furthermore, for the same time-horizon, the VaR calculations for both the 95% and the 97.5%
confidence level do not pick up at all any microstructural interdependencies.
B.3. Importance of Feedback Effects

To clarify the impact of feedback effects, we study the difference between risk figures we obtain by a first-order approximation with the risk figures from the full model. This difference captures all additional risk caused by feedback and looping among the firms within the graph.

Table 9 reports the percentage increase in VaR and expected shortfall caused by feedback effects for different confidence levels, i.e., we calculate \( \frac{\text{VaR}(\infty) - \text{VaR}(1)}{\text{VaR}(\infty)} \) in percentage numbers for the VaR and \( \frac{\text{ES}(\infty) - \text{ES}(1)}{\text{ES}(\infty)} \) in percentage numbers for the expected shortfall (ES).

In Table 9 we see that feedback effects are tail effects, since they heavily increase the tails of the distribution. For a one-year horizon and a high initial portfolio rating, the influence of feedback effects from microstructural interdependencies on the VaR is zero for all confidence levels below 97.5%. Only for a 99% confidence level, do we find an increase in VaR of around 25%. For the 95% confidence level, there is an increase in VaR caused by feedback effects only after a three-year horizon. However, if we measure risk as expected shortfall, we see an increase of more than 25% in risk for the one-year time horizon and the 95% confidence bound. This finding follows from the fact that, contrary to VaR, expected shortfall takes into account all tail events. Therefore, the expected shortfall gives a better picture of the portfolio’s true tail risk.

B.4. Marginal Risk Contribution

From a practical viewpoint, it is of paramount importance to quantify the marginal risk from adding another counterparty to the bank’s credit portfolio.

To analyze marginal risk contributions in the RBP, we consider the marginal contributions of five representative debtors. Debtor 1 is a large real estate firm, debtor 2 an energy holding, debtor 3 a large bank holding, debtor 4 a large retail company, and debtor 5 is a small debtor with only small microstructural interdependence. Thus, we consider debtors that differ in how
they are embedded into the microstructural topology of the credit portfolio. For our analysis, we assume that debtors 3 and 4 share the same sector-specific risk factor.

Figure 9 plots the marginal risk contributions for one- and three-year time horizons. We use a sample size of $10^9$ simulations and calculate both the additional VaR and expected shortfall that result from adding one of the debtors to the bank’s debt portfolio. The y-axis plots the additional risk per dollar of the credit portfolio’s notional as a function of the confidence level. A negative additional risk means that by adding the corresponding debtor, the credit risk manager can exploit some diversification effect, since one dollar of the credit portfolio will bear less risk than before. (We note that this decrease in relative risk does not mean that the absolute risk level of the credit portfolio becomes lower.) However, as we see from Figure 9, adding another debtor does not always result in a better diversification. Indeed, the existence of microstructural interdependence may lead to antidiversification and to an increase in the relative portfolio risk (expressed as risk per dollar). As we expected, the contribution of adding a small debtor to the portfolio (debtor 5) is almost zero.

For the one-year horizon, a striking property of our calculations is that the marginal VaR for debtors 1 through 5 does not differ significantly below the 98% confidence level. However, the marginal expected shortfall differs at the 95% confidence level. Again, we attribute this salient feature to the fact that microstructural interdependence acts on the tails. This finding becomes most evident for debtor 3. Debtor 3’s VaR dramatically increases beyond the 98% confidence level. We anticipate this increase in the expected shortfall with lower confidence levels. Therefore, even an expected shortfall with a low confidence level gives a reliable estimate of tail risks.

Marginal VaR and expected shortfall need not be order preserving. For example, consider the 95% confidence level as the relevant confidence level. At this level, if we define risk as VaR, debtor 3 seems to lower the risk per dollar. However, if we consider expected shortfall, debtor 3 increases the portfolio’s risk per dollar. For a confidence level of 97%, the opposite conclusion would hold if we analyze debtor 4. Therefore, VaR leads to a different assessment of a debtor’s marginal risk contribution compared to the expected shortfall. Therefore, VaR gives the wrong guidance for the risk management of loan portfolios.
When we consider a time horizon of three years, we see that the microstructural interdependencies not only influence the very outer tails, but their influence also seems to spread across lower confidence levels such that VaR better picks up the loss distribution. The difference between the expected shortfall and VaR becomes less severe. However, when we compare risk figures below the 97% confidence level, we still observe a large bias for debtor 2. Therefore, even for longer-term horizons we suggest to use expected shortfall instead of VaR.

If we only used a pure macroeconomic model, what would the marginal risk contributions look like? Figure 10 plots the same numbers as in Figure 9, but in Figure 10 we base all calculations on a pure macroeconomic model. Since microstructural interdependencies are absent, VaR and expected shortfall are almost the same. For the one-year time horizon, almost all debtors induce a lowering of the relative risk level, and therefore seem to provide a potential diversification effect. However, the effects are rather small. Interestingly, when we compare the Figure 10 VaR with the one-year VaR in Figure 9, which includes microstructural effects, there is no significant difference below the 97% confidence level. Hence, considering VaR with a low confidence level as the relevant risk measure turns out to be as misleading as discarding microstructural interdependencies.

For a time horizon of three years, these effects become slightly larger. At least for some confidence levels, debtors 3 and 4 contribute positively to the risk per dollar. Since these two debtors belong to the same sector, they provide exactly the same marginal risk in the macroeconomic model. Thus, if a bank bases credit decision on a purely macroeconomic model, these debtors would be treated as perfect substitutes. However, if we return to the microstructural model, the picture changes radically. We see, it is only debtor 3 who contributes to an increase in the relative risk. Debtor 4 provides a diversification opportunity. For the three-year time horizon, neglecting the microstructural interdependence weighs more heavily for lower confidence levels. The macroeconomic model underestimates the marginal expected shortfall for debtor 3 by at least a factor of ten. Also, the diversification potential of debtor 4 remains undetected.
V. Conclusion

To evaluate and manage a bank’s credit risk, it is not sufficient to scrutinize individual debtors. We must identify the concentration of risk within the credit portfolio. Such concentration arises not only in portfolios which are poorly diversified by sectors, but more importantly, also in portfolios that exhibit microstructural dependencies. Our model of credit contagion shows that microstructural interdependencies dramatically change the tail behavior of portfolio credit losses. The tail is the part of the distribution that both banks and regulators are most concerned about.

We find that using a purely macroeconomic model holds substantial model risk. A medium-sized credit portfolio with a well-diversified microstructure exhibits a substantial increase in default risk. This underestimation increases further when we consider a hierarchical interdependence structure.

However, of greater concern is our finding for the real business portfolio. This portfolio consists of subportfolios with random and hierarchical microstructures. The fact that large debtors link these subportfolios through their business interdependencies leads to a boosting of the loss distribution’s tails. In large part, this boost is induced by feedback effects.

Since microstructural interdependencies act only on the very far tails, using VaR to calculate credit risk, as proposed by regulators, leads to a dangerous underestimation of portfolio losses. Indeed, depending on the confidence levels, when we include microstructural interdependencies, the resulting VaR may not differ from a pure macrostructural model.

Finally, using VaR is misleading for capital allocation. Therefore, it is important to use a risk measure that takes into account the tails of the portfolio’s loss distribution. Expected shortfall is such a risk measure.

Based on our modeling approach, a promising direction for future research is to explore the implications of microstructural dependencies on the pricing of credit-portfolio derivatives.
Appendix: Proof of Proposition 2

Consider the $n$th-order approximation of the return $A$,

$$A^{(n)} = D_Y^{(n)} D \left( \sqrt{1 - v^2} \right) Y + D_\epsilon^{(n)} D (v) \varepsilon^{(n)} (G, Y, \epsilon),$$  \hfill (A.1)

where $\varepsilon^{(0)} (G, Y, \epsilon) = \epsilon$ and $D_Y^{(0)} = D_\epsilon^{(0)} = 1$. The idiosyncratic part is given as

$$\varepsilon^{(n)} (G, Y, \epsilon) = \Xi A^{(n-1)} + D (\eta) \epsilon.$$  \hfill (A.2)

Let

$$\varepsilon^{(n)} (G, Y, \epsilon) = E_Y^{(n)} Y + E_\epsilon^{(n)} \epsilon.$$  \hfill (A.3)

The terms $E_Y^{(n)}$ and $E_\epsilon^{(n)}$ must be determined recursively by using (A.2). Consider, e.g., the second-order approximation $A^{(2)}$. We obtain

$$\varepsilon^{(2)} (G, Y, \epsilon) = \Xi A^{(1)} + D (\eta) \epsilon \quad \hfill (A.4)$$

$$= \Xi \left( D_Y^{(1)} D \left( \sqrt{1 - v^2} \right) Y + D_\epsilon^{(1)} D (v) \varepsilon^{(1)} (G, Y, \epsilon) \right) + D (\eta) \epsilon \quad \hfill (A.5)$$

$$= \Xi \left( D_Y^{(1)} D \left( \sqrt{1 - v^2} \right) Y + D_\epsilon^{(1)} D (v) \left( E_Y^{(1)} Y + E_\epsilon^{(1)} \epsilon \right) \right) + D (\eta) \epsilon. \quad \hfill (A.6)$$

Then,

$$E_Y^{(2)} = \Xi D_Y^{(1)} D \left( \sqrt{1 - v^2} \right) + D_\epsilon^{(1)} D (v) E_Y^{(1)} \quad \hfill (A.7)$$

$$= \Xi D_Y^{(1)} D \left( \sqrt{1 - v^2} \right) + D_\epsilon^{(1)} D (v) \Xi D_Y^{(0)} D \left( \sqrt{1 - v^2} \right), \quad \hfill (A.8)$$

$$E_\epsilon^{(2)} = \Xi D_\epsilon^{(1)} D (v) E_\epsilon^{(1)} + D (\eta) \quad \hfill (A.9)$$

$$= \Xi D_\epsilon^{(1)} D (v) \left( \Xi D_\epsilon^{(0)} D (v) + D (\eta) \right) + D (\eta). \quad \hfill (A.10)$$

Therefore, the second-order return is

$$A^{(2)} = \left( D_Y^{(2)} D \left( \sqrt{1 - v^2} \right) + D_\epsilon^{(2)} D (v) E_Y^{(2)} \right) Y + D_\epsilon^{(2)} D (v) E_\epsilon^{(2)} \epsilon. \quad \hfill (A.11)$$

We still must determine the normalizing matrices $D_Y^{(2)}$ and $D_\epsilon^{(2)}$. We can proceed as for the first-order approximation, but with the appropriate adjustments, i.e., $E_Y^{(1)} \rightarrow E_Y^{(2)}$ and $E_\epsilon^{(1)} \rightarrow E_\epsilon^{(2)}$, respectively.
Then, it is straightforward to obtain the second-order normalizing matrices that guarantee consistency with the conditions (C1) and (C2),

\[
\begin{align*}
(C1) & : \quad (D(2)_ε)^2 = D\left(\Sigma^{(2)}\right)^{-1}, \\
(C2) & : \quad 1 = Q(D_Y^{(2)}, D_ε^{(2)}). 
\end{align*}
\] (A.12)

We note that compared to the first-order approximation, the second-order approximation already allows for feedback or back-propagation effects.

The third-order approximation for \(ε^{(3)}(G, Y, \epsilon)\) is easily obtained as

\[
E_Y^{(3)} = \Xi \left( D_Y^{(2)} D \left( \sqrt{1 - v^2} \right) + D_ε^{(2)} D (v) E_Y^{(2)} \right) \] (A.13)
\[
= \Xi D_Y^{(2)} D (v) E_Y^{(2)} + D (\eta) \] (A.15)
\[
= \Xi D_ε^{(2)} D (v) \Xi D_Y^{(1)} D (v) \Xi D_Y^{(0)} D (v) D^2 \left( \sqrt{1 - v^2} \right) \] (A.16)
\[
+ \Xi D_ε^{(2)} D (v) D (\eta) + D (\eta). \] (A.17)

We obtain the higher-order approximations in the same manner by recursively replacing \(E_Y^{(n)}\) and \(E_ε^{(n)}\) in the expression for \(ε^{(n)}(G, Y, \epsilon)\) by

\[
E_Y^{(n+1)} = \sum_{i=0}^{n} \left( \prod_{j=0}^{n-i-1} \Xi D_ε^{(n-j)} D (v) \right) \Xi D_Y^{(i)} D \left( \sqrt{1 - v^2} \right), \] (A.18)
\[
E_ε^{(n+1)} = \sum_{i=0}^{n} \left( \prod_{j=0}^{n-i-1} \Xi D_ε^{(n-j)} D (v) \right) D (\eta) + \sum_{i=0}^{n} \Xi D_ε^{(n-i)} D (v). \] (A.19)

By doing so, we obtain \(ε^{(n+1)}(G, Y, \epsilon)\), and by following the same arguments as above, the normalizing matrices \(D_Y^{(n+1)}\) and \(D_ε^{(n+1)}\).
Figure 1. Integrating macro- with microstructural effects. Panel A illustrates the interdependencies in a pure macrostructural model. Dependency of default is solely generated through $\Lambda$. In Panel B, we add a microstructural channel that allows for business interdependencies. The idiosyncratic risk of firm $j$ has now an impact on firm $i$. In addition, the macrostructural variable $Z_{s(j)}$ can now also influence debtor $i$ through her business relation with debtor $j$. 

32
Figure 2. Heavy Gravity Portfolio (HGP). In this hierarchical portfolio structure, two firms have a major business impact on their direct suppliers and an indirect one to the second-level suppliers. We assume that the two large firms are weakly interdependent.
Figure 3. Diversified Debtor’s Portfolio (DDP). The microstructure is generated from a random graph. We randomly select edges with probability $p = 102/(51 \times 101)$ to obtain an average of 102 edges. Contrary to the HGP case, there is no firm in the portfolio that is an obvious center of business gravity. The firms on the top right-hand side of the figure have no microstructural interdependence.
Figure 4. Additional correlation in the DDP between the synthetic returns $A$. The figure plots the contours of the additional correlation caused by microstructural interdependencies in the DDP compared to the pure macroeconomic model. The additional correlation is expressed as the absolute difference between the model including microstructural effects and the pure macroeconomic model. The first-order approximation cannot capture microstructural feedback effects. Panel B illustrates the additional correlation in the full model, taking feedback effects into account.
Figure 5. Additional correlation in the HGP between the synthetic returns $A$. The figure plots the additional correlation caused by microstructural interdependencies in the HGP compared to the pure macroeconomic model. The additional correlation is expressed as the absolute difference between the model including microstructural effects and the pure macroeconomic model. Panel A plots the first-order additional correlation. The first-order approximation cannot capture microstructural feedback effects. Panel B illustrates the additional correlation in the full model, taking feedback effects into account.
Figure 6. Additional correlation in the RBP between the synthetic returns $A$. The figure plots the additional correlation caused by microstructural interdependencies in the RBP compared to the pure macroeconomic model. The additional correlation is expressed as the absolute difference between the model including microstructural effects and the pure macroeconomic model. Panel A plots the first-order additional correlation. The first-order approximation cannot capture microstructural feedback effects. Panel B illustrates the additional correlation in the full model, taking feedback effects into account.
Figure 7. Rating volatilities of the DDP and HGP. In each panel, the lower surface corresponds to rating-count volatility in a pure macroeconomic model. The upper surface plots the rating-count volatility for the model that includes microstructural effects. Panels A and B show the rating volatilities of the HGP as a function of the rating class and time horizon. Panel A assumes an initial portfolio with high rating quality and Panel B with a very low rating quality. Panels C and D show the rating-count volatility for the DDP. Panel C assumes a high initial rating quality, Panel D a very low initial rating quality.
Figure 8. Rating volatilities of the RBP. The panels show the rating volatilities of the RBP as a function of rating class and time horizon. In each panel, the lower surface corresponds to rating-count volatility in a pure macroeconomic model. The upper surface plots the rating-count volatility for the model that includes microstructural effects. Panel A assumes an initial portfolio with high rating quality and Panel B assumes an initial portfolio with a very low rating quality.
Figure 9. Marginal risk contribution in the RBP with microstructural interdependencies. The panels show the marginal risk contribution of different debtors for different time horizons and different risk measures. The marginal risk is expressed as additional risk per dollar of the credit portfolio’s notional. We calculate the risk figures for different confidence levels based on a sample of $10^9$ simulations.
Figure 10. Marginal risk contribution in the RBP without microstructural interdependencies. The panels show the marginal risk contribution of different debtors for different time horizons and different risk measures. The marginal risk is expressed as additional risk per dollar of the credit portfolio’s notional. We calculate the risk figures for different confidence levels based on a sample of $10^9$ simulations.
The sector classification corresponds to BAK data. The first two columns are the sector classifications. Column three and four indicate, in absolute and relative terms, how many debtors we assume in each sector for our test portfolios. The total amount of debtors is 102. In the last column, we calculate the sector specific weights $w_s$ by using the calibration procedure outlined in Section [III].

<table>
<thead>
<tr>
<th>Sector</th>
<th>Industry</th>
<th>Debtors</th>
<th>%</th>
<th>$w_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Food, Agriculture</td>
<td>8</td>
<td>7.84</td>
<td>0.9853</td>
</tr>
<tr>
<td>2</td>
<td>Textiles</td>
<td>6</td>
<td>5.88</td>
<td>0.9758</td>
</tr>
<tr>
<td>3</td>
<td>Paper and Printing, Wood Manufacturing</td>
<td>7</td>
<td>6.86</td>
<td>0.9773</td>
</tr>
<tr>
<td>4</td>
<td>Chemical</td>
<td>4</td>
<td>3.92</td>
<td>0.9838</td>
</tr>
<tr>
<td>5</td>
<td>Plastics</td>
<td>5</td>
<td>4.90</td>
<td>0.9779</td>
</tr>
<tr>
<td>6</td>
<td>Metal and Machine</td>
<td>9</td>
<td>8.82</td>
<td>0.9922</td>
</tr>
<tr>
<td>7</td>
<td>Electronics</td>
<td>8</td>
<td>7.84</td>
<td>0.9926</td>
</tr>
<tr>
<td>8</td>
<td>Other Manufacturing, Recycling</td>
<td>6</td>
<td>5.88</td>
<td>0.9781</td>
</tr>
<tr>
<td>9</td>
<td>Construction</td>
<td>12</td>
<td>11.76</td>
<td>0.9870</td>
</tr>
<tr>
<td>10</td>
<td>Retail, Commerce</td>
<td>6</td>
<td>5.88</td>
<td>0.9834</td>
</tr>
<tr>
<td>11</td>
<td>Hotel and Restaurant</td>
<td>9</td>
<td>8.82</td>
<td>0.9603</td>
</tr>
<tr>
<td>12</td>
<td>Traffics, Communication, Energy, Water</td>
<td>9</td>
<td>8.82</td>
<td>0.9787</td>
</tr>
<tr>
<td>13</td>
<td>Finance and Insurance</td>
<td>7</td>
<td>6.86</td>
<td>0.9818</td>
</tr>
<tr>
<td>14</td>
<td>Real Estate</td>
<td>6</td>
<td>5.88</td>
<td>0.9927</td>
</tr>
</tbody>
</table>
We adapt the migration matrix from Moody’s migration matrix based on historical default data between 1970 and 2002. Migration occurs from rows to columns, i.e., entries denote the probabilities that a firm’s rating moves from row $d = 1, \ldots, 8$ to column $d = 1, \ldots, 8$.

<table>
<thead>
<tr>
<th>$d$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9181</td>
<td>0.0741</td>
<td>0.0076</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0001</td>
</tr>
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<td>0.0119</td>
<td>0.9084</td>
<td>0.0759</td>
<td>0.0027</td>
<td>0.0008</td>
<td>0.0001</td>
<td>0.0000</td>
<td>0.0002</td>
</tr>
<tr>
<td>3</td>
<td>0.0005</td>
<td>0.0240</td>
<td>0.9189</td>
<td>0.0499</td>
<td>0.0051</td>
<td>0.0013</td>
<td>0.0001</td>
<td>0.0002</td>
</tr>
<tr>
<td>4</td>
<td>0.0005</td>
<td>0.0025</td>
<td>0.0533</td>
<td>0.8839</td>
<td>0.0487</td>
<td>0.0077</td>
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<td>0.0018</td>
</tr>
<tr>
<td>5</td>
<td>0.0001</td>
<td>0.0005</td>
<td>0.0050</td>
<td>0.0552</td>
<td>0.8517</td>
<td>0.0696</td>
<td>0.0051</td>
<td>0.0128</td>
</tr>
<tr>
<td>6</td>
<td>0.0001</td>
<td>0.0003</td>
<td>0.0014</td>
<td>0.0044</td>
<td>0.0660</td>
<td>0.8315</td>
<td>0.0288</td>
<td>0.0675</td>
</tr>
<tr>
<td>7</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0059</td>
<td>0.0178</td>
<td>0.0413</td>
<td>0.6799</td>
<td>0.2550</td>
</tr>
<tr>
<td>8</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
For our test portfolio, we consider four different initial credit qualities. Entries are percentage numbers of debtors in each rating grade according to Gordy (2000). These rating compositions are representative for typical credit portfolios found in the banking industry.

<table>
<thead>
<tr>
<th>Rating Grade</th>
<th>Portfolio Credit Quality</th>
</tr>
</thead>
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<td></td>
<td>High</td>
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<td>1</td>
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<tr>
<td>2</td>
<td>5.90</td>
</tr>
<tr>
<td>3</td>
<td>29.26</td>
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<tr>
<td>4</td>
<td>37.92</td>
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<td>5</td>
<td>19.08</td>
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<tr>
<td>6</td>
<td>2.72</td>
</tr>
<tr>
<td>7</td>
<td>1.30</td>
</tr>
</tbody>
</table>
Given the default rates and the migration matrix in Table 2, we obtain the threshold values by calibrating to the macroeconomic data as outlined in Section III. Given a firm $i$, a credit migration from a rating class $x$ to a new class $y = 1, \ldots, 9$ is triggered whenever the return $A_i$ exceeds the threshold value $\theta_{xy}$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\theta_{x0}$</th>
<th>$\theta_{x8}$</th>
<th>$\theta_{x7}$</th>
<th>$\theta_{x6}$</th>
<th>$\theta_{x5}$</th>
<th>$\theta_{x4}$</th>
<th>$\theta_{x3}$</th>
<th>$\theta_{x2}$</th>
<th>$\theta_{x1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>-3.7190</td>
<td>-3.7190</td>
<td>-3.7190</td>
<td>-3.7190</td>
<td>-2.4221</td>
<td>-1.3927</td>
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<td>$\infty$</td>
</tr>
<tr>
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<td>$-\infty$</td>
<td>-3.5329</td>
<td>-3.5329</td>
<td>-3.4243</td>
<td>-3.0537</td>
<td>-2.6692</td>
<td>-1.4069</td>
<td>2.2598</td>
<td>$\infty$</td>
</tr>
<tr>
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<td>$-\infty$</td>
<td>-3.5317</td>
<td>-3.4230</td>
<td>-2.9381</td>
<td>-2.4725</td>
<td>-1.5843</td>
<td>1.9675</td>
<td>3.2816</td>
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</tr>
<tr>
<td>4</td>
<td>$-\infty$</td>
<td>-2.9147</td>
<td>-2.7114</td>
<td>-2.2870</td>
<td>-1.5566</td>
<td>1.5864</td>
<td>2.7438</td>
<td>3.2776</td>
<td>$\infty$</td>
</tr>
<tr>
<td>5</td>
<td>$-\infty$</td>
<td>-2.2319</td>
<td>-2.0998</td>
<td>-1.3568</td>
<td>1.5478</td>
<td>2.5365</td>
<td>3.2178</td>
<td>3.7003</td>
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<tr>
<td>6</td>
<td>$-\infty$</td>
<td>-1.4947</td>
<td>-1.3028</td>
<td>1.4595</td>
<td>2.4997</td>
<td>2.9075</td>
<td>3.3336</td>
<td>3.7015</td>
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</tr>
<tr>
<td>7</td>
<td>$-\infty$</td>
<td>0.6588</td>
<td>1.5133</td>
<td>1.9811</td>
<td>2.5152</td>
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<td>$\infty$</td>
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<tr>
<td>8</td>
<td>$-\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>
Given the sector risk weights $w_s$ in Table 1 and threshold values $\theta_{xy}$ in Table 4, we can calibrate the sector correlations $\rho_{s(i),s(j)}$, $i, j = 1, \ldots, 14$, to the historical default data between 1980 and 1997 by using the procedure outlined in Section III.

### Table 5

Sector correlation matrix $\Lambda$

<table>
<thead>
<tr>
<th>Sector</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>0.82</td>
<td>0.81</td>
<td>0.76</td>
<td>0.47</td>
<td>0.76</td>
<td>0.80</td>
<td>0.84</td>
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<td>0.79</td>
<td>0.80</td>
<td>0.70</td>
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<tr>
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<td>0.94</td>
<td>0.93</td>
<td>0.84</td>
<td>0.93</td>
<td>0.84</td>
<td>0.93</td>
<td>0.91</td>
<td>0.56</td>
<td></td>
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<td>0.61</td>
<td>0.69</td>
<td>0.73</td>
<td>0.73</td>
<td>0.47</td>
<td>0.39</td>
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<td>1.00</td>
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<td>0.96</td>
<td>0.66</td>
<td>0.82</td>
<td>0.96</td>
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<tr>
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<td>0.94</td>
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<td>0.61</td>
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</tr>
</tbody>
</table>
### Table 6

**Additional Risk in the DDP**

Entries report the percentage increase in VaR and expected shortfall for default losses caused by microstructural interdependencies in the DDP. We analyze different confidence levels and different time horizons. The results are based on a simulation with $10^6$ samples.

### Panel A: high initial portfolio rating quality

<table>
<thead>
<tr>
<th>confidence</th>
<th>time (yrs.)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>VaR</th>
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Entries report the percentage increase in VaR and expected shortfall for default losses caused by microstructural interdependencies in the HGP. We analyze different confidence levels and different time horizons. The results are based on a simulation with $10^6$ samples.

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Entries report the percentage increase in VaR and expected shortfall for default losses caused by microstructural interdependencies in the RBP. We analyze different confidence levels and different time horizons. The results are based on a simulation with $10^6$ samples.

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Table 9  
Importance of Feedback Effects for the RBP

Entries report the percentage increase in VaR and expected shortfall for default losses caused by feedback effect. We analyze different confidence levels and different time horizons. The results are based on a simulation with $10^6$ samples.

**Panel A: high initial portfolio rating quality**

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<td>95.0</td>
<td>84.1</td>
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**Panel B: average initial portfolio rating quality**

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<td>45.5</td>
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**Panel C: low initial portfolio rating quality**

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<tbody>
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**Panel D: very low initial portfolio rating quality**

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<td>67.2</td>
<td>55.6</td>
<td>42.1</td>
<td>33.3</td>
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</tbody>
</table>
Notes

1See, e.g., Fons (1991) and Jonsson and Fridson (1996).


3We note that the business matrix can also be used to capture the impact of different initial ratings. As an example, a firm $j$’s shock might influence another firm’s return to a lesser extent if firm $j$ has a high rating compared to the case if firm $j$ were in a low rating category.

4The derivation relies on nested Neumann series. For more details on the mathematical properties of the model, we refer to the technical paper by Egloff and Leippold (2003).

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6Note that if such a firm is not part of the bank’s credit portfolio, the firm still has an effect on the bank’s credit risk through the microstructural dependencies with the bank’s debtors.

7Here, $\prod$ is understood as the matrix product operator.
References


Yu, F.: 2005, Correlated default in reduced-form models, \emph{Mathematical Finance} \textbf{forthcoming}.