swissQuant
Mastering Complexity

Mathematical Challenge June 2017

Optimal Portfolio Distribution

References

- [1] C. M. Jarque and A. K. Bera. "A Test for Normality of Observations and Regression Residuals"
- [2] A. E. Raftery, D. Madigan, J. A. Hoeting. "Bayesian Model Averaging for Linear Regression Models"
- [3] E. I. George and R. E. McChulloch. "Approaches for Bayesian Variable Selection"

Description

Introduction

Many mathematical procedures allow to find exposure coefficients (called *betas* [1]) of a financial instrument respectively to a selection of risk factors. Whereas most of these techniques provide point-estimate betas, the Bayesian Model Averaging (BMA) framework allows to compute beta distributions. We first briefly describe the formulas and procedure underneath these distributions. We then state the driven portfolio optimization problem.

Formulation

When implementing a model to explain financial returns, we subjectively chose a set of factors that seem to fit our data. Hence at the moment this set is built, we incorporate our beliefs as we define the model risk factor set.

What about model uncertainty? Roughly speaking, the BMA framework allows us to take into account this model risk by considering the set of possible models denoted by $\{M_\gamma\}_\gamma$. We can then aggregate the information from all these models. The outcome of BMA takes the form of beta distributions, where the previously mentioned uncertainty is represented.

Technical Details

We define instrument returns y for model M_γ as

$$y|M_\gamma \sim N(X_\gamma \beta_\gamma, \sigma^2 I),$$

where $X\gamma$ is the return matrix of risk factors composing the model M_γ.

In order to combine the different models' information, we need to compute their respective model probabilities $P(M_\gamma|y)$.

Using Bayes' formula,

$$P(M_\gamma|y_{obs}) \propto P(y_{obs}|M_\gamma)P(M_\gamma)$$

where

$$P(y|M_\gamma) = \iint P(y|\beta_\gamma, \sigma^2, M\gamma)P(\beta_\gamma|\sigma^2, M\gamma)P(\sigma|M\gamma)\mathrm{d}\beta_\gamma \mathrm{d}(\sigma^2). \quad \textbf{(1)}$$

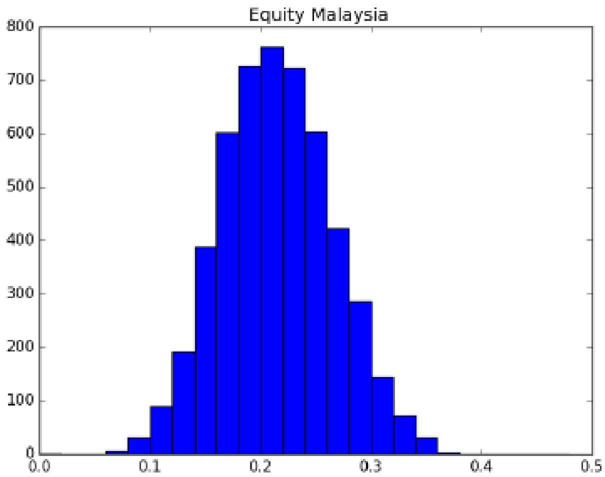
With a suited choice of priors [2], a closed-form formula can be computed.

Beta Distributions

A beta distribution observation is available for each model M_γ. We compute these observations using the model probability $P(M\gamma|y)$ and its respective OLS beta estimation $\beta_{\gamma,\,\mathrm{OLS}}$ with:

$$P(M\gamma|y) \cdot \beta_{\gamma,\,\mathrm{OLS}}$$

An empirical distribution is obtained by gathering the observations. Below can be seen an example of a Malaysia equity beta distribution on some Asian-driven strategy.



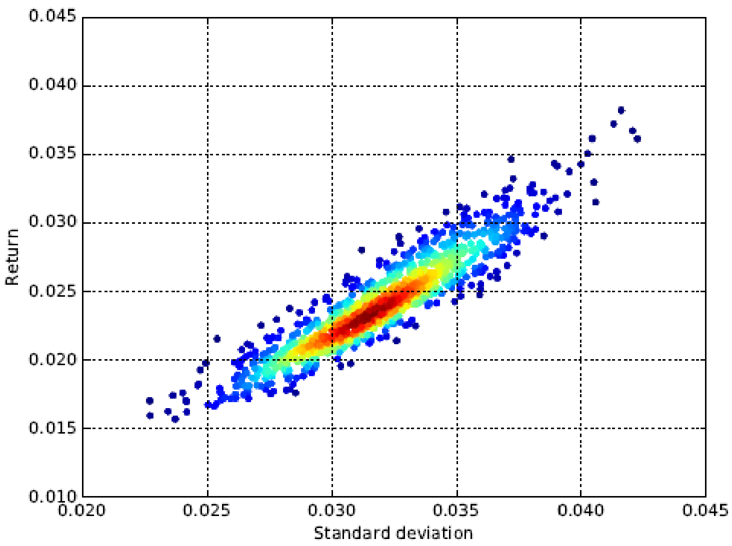
Optimal Portfolio Distribution

Let's consider N assets in the market from which we construct our optimal portfolio. We call μ_N and Σ_N the mean vector and covariance matrix of these assets returns, respectively. These are obtained using

$$\mu_N = B\mu_{riskFactors}$$
$$\Sigma_N = B\Sigma_{riskFactors}B'$$

where B is the beta matrix.

As an example, we consider the Mean-Variance optimization problem. Let's sample point-estimate betas from their distributions. Corresponding "sample" optimal portfolios can thus be computed, which can be represented by the "portfolio cloud" below.



Questions:

- Q1. Compute the closed-form formula of the model probability $P(y|M_\gamma)$ given by **(1)** using the priors specified in [2].
- Q2. Analyze how does the optimal allocation driven by the Bayesian predictive distribution $P(y|y_{obs}) = \Sigma_\gamma P(y|M_\gamma)P(M_\gamma|y_{obs})$ differ from the ones obtained using the Maximum Likelihood and the Maximum a Posteriori point-estimates.
- Q3. The obtained cloud of portfolios gives us relevant insights on the model uncertainty. However, it is not clear how this uncertainty affects the rebalancing. Indeed, we need to measure how close our current portfolio is from the optimum to deduct if rebalancing is indeed necessary. Moreover, rebalancing always causes additional costs that may cancel out gains from position changes. Suggest a suitable metric/algorithm to prevent ineffective rebalancing.

We look forward to your opinions and insights. Diane Thizeau is responsible for the distribution of all results to the Methodology Board.

Best Regards,

swissQuant Group Leadership Team