swissQuant
Mastering Complexity

Mathematical Challenge November 2019

Optimal Execution of Trades

References

[1] A. Cartea and S. Jaimungal. Incorporating order-flow into optimal execution. *Math. Finan. Econ.*, 10:339–364, 2016.

[2] R. Almgren and N. Chriss. Optimal execution of portfolio transactions. *J. Risk*, pages 5–40, 2001.

[3] A. Cartea, S. Jaimungal, and J. Penalva. *Algorithmic and High-Frequency Trading*. Cambridge University Press, 2015.

Description

Motivation

Institutional investors who execute large trades in the market experience direct and indirect costs. Direct costs are known upfront and are easy to compute. Examples include fees and other transaction costs which need to be paid to brokers and exchanges in order to trade. Indirect costs are costs arising from the price impact that the trader's actions cause in the market. Such costs can be difficult to quantify even after the order has been executed. In most cases a trader's actions move the market against the trader, meaning that the price impact has a negative effect on the execution price obtained. When liquidating a position price impact causes lower revenues, whereas it results in higher prices when shares are being bought [1].
The field of optimal execution is concerned with minimizing the indirect costs of trading. Typically a performance criterion is defined and optimal control techniques are applied to find the trading strategy which maximises the expected value of this criterion [2].

Technical Details

Consider an agent liquidating N shares over a time horizon of T using market orders. The agent's current inventory, Q_t^v, is described by

$$dQ_t^v = -v_t dt, \quad Q_0^v = N. \tag{1}$$

Here v_t is the speed at which the agent liquidates shares. The mid-price of the stock being traded, S_t, is assumed to follow the stochastic differential equation

$$dS_t^v = b(\mu_t - v_t)dt + \sigma dW_t, \tag{2}$$

where b is a positive constant describing the permanent price impact caused by market orders. μ_t is the net order flow (buy orders minus sell orders) caused by other market participants and W is a standard Brownian motion and is independent of μ_t. Because only a limited number of buy or sell offers exist at the best price, large market orders will walk the book. Consequently, the actual execution price achieved by the agent will be worse than the current bid. To model this, a tempory price impact is introduced into the model such that the average execution price is given by

$$\hat{S}_t^v = S_t^v - \left(kv_t + \frac{1}{2}\Delta\right), \tag{3}$$

where Δ is the bid-ask spread and k is the non-negative, constant temporary price impact parameter.
Define the performance criterion to be

$$H^v = \mathbb{E}\left[X_T + Q_T^v(S_T^v - \frac{1}{2}\Delta - \alpha Q_T^v) - \phi \int_t^T (Q_u^v)^2 du\right]. \tag{4}$$

Here $\alpha \geq 0$ is a parameter which penalises strategies which have a non-zero share inventory at time T, and $\phi \geq 0$ is an urgency parameter [3]. A larger value of ϕ leads to strategies which trade more quickly at the start of the trading period. X_t is the cash generated from selling shares and obeys

$$dX_t = \hat{S}_t^v v_t dt, \quad X_0 = 0. \tag{5}$$

Questions

- **Q1:** *Consider first the case that $\mu_t = 0$. Show that the trading strategy which maximises equation* (4) *is given by:*

$$v_t^* = \gamma \frac{\zeta e^{\gamma(T-t)} + e^{-\gamma(T-t)}}{\zeta e^{\gamma(T-t)} - e^{-\gamma(T-t)}} Q_t^{v^*}, \tag{6}$$

with the constants

$$\gamma = \sqrt{\frac{\phi}{k}} \quad \text{and} \quad \zeta = \frac{\alpha - \frac{b}{2} + \sqrt{k\phi}}{\alpha - \frac{b}{2} - \sqrt{k\phi}}.$$

Here $Q_t^{v^*}$ is the value of Q_t^v obtained by following the optimal strategy up to time t. By taking an appropriate limit, show that the optimal strategy which guarantees all shares are sold within the time horizon T is:

$$v_t^* = \gamma \frac{\cosh(\gamma(T-t))}{\sinh(\gamma(T-t))} Q_t^{v^*}. \tag{7}$$

Are any other limits of equation (6) *of practical interest? Are these limits in line with intuition regarding the optimal trading strategy?*

Hint: Solve the Hamilton-Jacobi-Bellman equation for the value function

$$H = \sup_v H^v. \tag{8}$$

- **Q2:** *Consider now $\mu_t \neq 0$. Show that the trading strategy which maximises equation* (4) *and guarantees all shares are sold within the time horizon T is given by:*

$$v_t^* = \gamma \frac{\cosh(\gamma(T-t))}{\sinh(\gamma(T-t))} Q_t^{v^*} - \frac{b}{2k} \int_t^T \frac{\sinh(\gamma(T-u))}{\sinh(\gamma(T-t))} \mathbb{E}[\mu_u | \mathcal{F}_t^\mu] du, \tag{9}$$

where $\mathcal{F}_t^\mu$ is the natural filtration generated by μ.

- **Q3:** *Comparing equations* (7) *and* (9)*, we observe that setting $\mu_t \neq 0$ introduces an extra term to the optimal strategy. What is the financial interpretation of this correction term? Is the behaviour of this term in line with financial intuition? Discuss whether equation* (9) *is a valid solution to the financial problem that we initially posed.*

We look forward to your opinions and insights.

Best Quant Regards,

swissQuant Group Leadership Team