Mathematical Challenge January 2018

Spread option pricing via Fast Fourier Transform

References

- [1] Carmona, R. and V. Durrleman (2003). Pricing and Hedging Spread Options. *SIAM Review*, 627-686
- [2] Carr, P. and D. Madan (1999). Option Valuation using the Fast Fourier Transform. *Journal of computational finance,* 2, 61-73.
- [3] Margrabe, W. (1978). Value of an Option to Exchange One Asset for Another. *The Journal of Finance 33(1)*, 177-186
- [4] Hurd, T. and Z. Zhou (2010). A Fourier Transform Method for Spread Option Pricing. *SIAM Journal on Financial Mathematics,* 1, 142-157

Description

Introduction

The spread option is a common multi-asset financial derivative that is mostly traded OTC but for certain underlying assets also trades on exchanges. It is an option written on the difference of two underlying assets with values at time t of $S_{t,1}$ and $S_{t,2}$. The European-style option gives the holder the right to be paid $S_{T,1} - S_{T,2}$ at a fixed maturity time T. Furthermore, to exercise the option, the holder must pay the exercise price K. The payoff at time T can thus be written as

$$P_T = \left(S_{T,1} - S_{T,1} - K\right)^+.$$

These options are mostly written on commodities which requires any pricing framework to be flexible in terms of modelling assumptions. For most models of underlying asset prices there is no closed-form solution for the spread option price. The interested reader is referred to [1]. A mentionable special case is the exchange option which is a spread option with $K = 0$. Margrabe showed, that for two assets with jointly normally distributed returns, the price of an exchange option can be computed explicitly, see [3].

Application

The approach that we are about to present is able to price spread options for a vast range of asset pricing models and is based on two assumptions. First, the joint characteristic function of the log-prices of the underlying assets has to be known analytically and second, the characteristic function has to factorize in a certain way as is discussed below. These assumptions are satisfied in the Black

Scholes model, models with mean-reverting properties such as the Vasicek model as well as Lévy models like the Variance-Gamma process that allow for jumps in the underlying asset price evolution.

Another advantage of this Fourier transform approach is that the Greeks can be computed with only marginally higher complexity (see Q3).

Formulation

In the following we restrict ourselves to the case of strike price $K = 1$ (see Q1).

Hurd and Zhou [4] showed that in that case, the option price can be expressed as:

$$\Pi(S_0) = e^{-rT}\boldsymbol{E}[(e^{X_{T,1}} - e^{X_{T,2}} - 1)^+] = \frac{e^{-rT}}{(2\pi)^2} \iint_{\mathbb{R}^2+i\epsilon} \boldsymbol{E}_{X_0}\left[e^{i\boldsymbol{u}'X_T}\right] \hat{P}(\boldsymbol{u})\, d^2\boldsymbol{u}$$

where $\epsilon = (\epsilon_1, \epsilon_2)'$ with $\epsilon_2 > 0$ and $\epsilon_1 + \epsilon_2 < -1$, $\boldsymbol{E}_{X_0}$ stands for the risk-neutral expectation with known initial log-asset prices X_0, $\Gamma(z)$ denotes the complex gamma function and

$$\hat{P}(\boldsymbol{u}) = \frac{\Gamma(i(u_1 + u_2) - 1) \cdot \Gamma(-iu_2)}{\Gamma(\mathrm{i}u_1 + 1)}.$$

In order to transform the above into a Fourier integral and subsequently apply FFT, Hurd and Zhou make the following modelling assumption:

> *Assumption (original):*
>
> *For any $t > 0$, the increment $X_t - X_0$ is independent of X_0. This implies that the characteristic function of X_T factorizes in $\boldsymbol{E}_{X_0}\left[e^{i\boldsymbol{u}'X_T}\right] = e^{i\boldsymbol{u}'X_0}\phi(\boldsymbol{u};T)$, where $\phi(\boldsymbol{u};T) = \boldsymbol{E}_{X_0}\left[e^{i\boldsymbol{u}'(X_T - X_0)}\right]$.*

It can be shown that this assumption is too restrictive for many practical applications. Spread options are often written on commodities which can exhibit mean-reverting properties. However, the above assumption does not hold for mean-reverting processes (see Q2). A reasonable modification of the original assumption is enough to extend the presented methodology also to mean-reverting processes.

> *Assumption (modified):*
>
> *For any $T > 0$, the characteristic function of X_T factorizes in $\boldsymbol{E}_{X_0}\left[e^{i\boldsymbol{u}'X_T}\right] = e^{i\boldsymbol{u}'V_0}\phi(\boldsymbol{u};T)$, where $\phi(u;T)$ is some function independent of X_0 and V_0 is a linear function of X_0.*

Under the modified assumption, we can finally express the spread option price as a Fourier Integral that can be solved efficiently through FFT. More details about general option valuation using FFT can be found in [2].

$$\Pi(S_0) = \frac{e^{-rT}}{(2\pi)^2} \iint_{\mathbb{R}^2} e^{i(\boldsymbol{u}+i\epsilon)'V_0}\phi(\boldsymbol{u} + i\epsilon; T)\hat{P}(\boldsymbol{u} + i\epsilon)\, d^2\boldsymbol{u} = e^{-rT}e^{-\epsilon' V_0} \cdot \mathrm{iFT2}[B]\big(V_0(X_0)\big),$$

where $B(\boldsymbol{u}) = \phi(\boldsymbol{u} + i\epsilon; T)\hat{P}(\boldsymbol{u} + i\epsilon)$ and $\mathrm{iFT2}[f](\omega)$ denotes the two dimensional inverse Fourier transform of the function f at ω.

Questions:

- Q1: Explain why it is sufficient to consider spread options with $K = 1$. For a rigorous proof, consider the cases of $K > 0$ and $K < 0$ separately.
- Q2: Show that for underlying assets under the log-Ornstein Uhlenbeck model, the original assumption is violated but the modified assumption is satisfied. Determine V_0.
- Q3: Write down the Fourier integral formulation for the option price's sensitivity to the first asset price Δ_1 making the Black Scholes assumption for the underlying assets. Are additional assumptions required?
- Q3: Implement a pricing script for a spread option, working for any $K \in \mathbb{R}$ (using the result of Q1) under the Black Scholes assumption. If you are using MATLAB, use the function *ifft2()* to compute the Fast Fourier Transform.