Mathematical Challenge February 2018

Value-at-Risk Estimation for long time horizons

References:

- [1] L. Spadafora, M. Dubrovich, M. Terraneo (2004), Value-at-Risk for long-term Risk Estimation, *The Journal of Risk*.
- [2] P. Embrechts, R. Kaufmann, P. Patie (2005), Strategic Long-Term Financial Risks: Single Risk Factors*, Computational Optimization and Applications 32(1-2),* 61-90.
- [3] J. Danielsson, J.P. Zigrand (2006), On Time.Scaling of Risk and Square-Root-of-Time Rule, *Journal of Banking and Finance 30(10),* 2701-2713.
- [4] D. Blake, A. Cairns, K. Dowd (2000), Extrapolating VaR by the SRR, *Financial Engineering News*.

Description:

In the aftermath of the financial crisis of 2008, the financial institutions had to revise their risk management policies and be compliant to stricter regulations. Typically, banks have to provide sufficient capital to cover several sources of risk. In light of the Basel III requirements, the regulatory capital is commonly estimated by the 99% Value-at-Risk (VaR) of the portfolio simulated P&L at time horizons under which the market positions can be hedged and liquidated. Such time horizons are short and a large panel of models on the returns distribution is available (assuming a parametric model or based on empirical data) to generate the risk scenarios necessary to the VaR calculation. The regulators require in addition to compute the so-called economic capital to cover the risk on longer time horizons that can be one or several years. The latter is achieved, by using one of the following approaches:

1. Generate the risk factor scenarios over the long time horizon and evaluate the portfolio.
2. Generate the risk factor scenarios for a horizon of one day[1], compute the corresponding VaR and then rescale it to longer time horizons.

The straightforward risk factor scenario generation in Approach 1 admits some limitations. First, it is assuming that the bank's positions are constant over the entire time horizon, which is definitely not the case in reality with regular rebalancing of the portfolio. Then, the amount of data to capture several economic cycles is limited making the non-parametric models non-applicable and the Monte-Carlo simulations have to rely on models where the parameters have to be wisely set. Due to such limitations, many practitioners prefer to use Approach 2 aiming to estimate the unknown function $h(.)$ such that $VaR_\alpha(T') = h\big(VaR_\alpha(T)\big)$, for $T' \neq T$. Under the assumption of normal distribution of the returns, the VaR scales up with the so-called Square-Root-of-Time Rule (SRTR), i.e.

[1] In [2], it is suggested to use higher horizons since the returns are dependent due to the volatility clustering.

$$VaR_\alpha(T') = \sqrt{\frac{T'}{T}} \cdot VaR_\alpha(T).$$

Unfortunately, the SRTR cannot be easily applicable to any kind of distribution. Typically, Danielsson studied the effect of the SRTR in the case of returns following a jump diffusion process and highlighted a downward bias in risk estimation (see [3]).

A VaR-scaling methodology is presented in [1], where the returns are assumed to be i.i.d. Student T and Variance-Gamma without drift. The VaR scaling is presented as a convolution problem, using the fact the Probability Density Function (PDF) of the sum of two i.i.d. random variables is given by:

$$p(y) = \int_{-\infty}^{+\infty} p(y - x_1(\Delta t) p\big(x_1(\Delta t)\big) dx_1(\Delta t),$$

where $y = x_1(\Delta t) + x_2(\Delta t)$, $x_i(\Delta t)$ denoting the returns of the portfolio for the time interval $\Delta t = t_i - t_{i-1}$. To get the PDF of P&L with a time horizon $T = n\Delta t$, we need to apply the convolution n times. In case of the normal distribution for the one-day returns, the convolution leads to the normal distribution with suitably rescaled mean and variance. The main findings of [1] are that if the short-term returns distribution has an exponential decay, then the SRTR is applicable, as the long-term returns distribution converges quickly to the normal distribution. However, if the short-term returns distribution has a power law decay, the SRTR is only applicable only if the tails are not too heavy. More precisely, the SRTR is applicable to the Variance-Gamma distribution according to the Central Limit Theorem (CLT). However, the Student T distribution converges to the normal distribution only for a specific range of ν which should be above a critical value determined as $\nu^* = 3.41$. Below this one, the CLT convergence is not ensured and the convolution has to be explicitly calculated.

Questions:

- **Q1:** Using the formulations' step similar to the ones introduced in [1], explain the scaling properties assuming that the short-term returns are i.i.d. distributed with zero drift according to the Generalized Pareto Distribution (GPD). Can the convolution explained in [1] be simplified in the case of the GPD?
- **Q2:** The VaR scaling procedure as a convolution problem relies on the independence of returns. Assuming a specific correlation structure between returns, is it possible to derive a semi-analytical scaling function for the VaR?
- **Q3:** For very long term risk horizons, the lack of data makes the backtesting of such methodology impossible. However in case of the availability of intraday data, make a study of backtesting performance of intraday data, rescaled daily, rescaled weekly and corresponding daily and weekly data. Do the rescaled daily/weekly and standard daily/weekly VaR highly differ?