Mathematical Challenge January 2019

Some properties of collision operators in kinetic theory

References

[1] C. Brugna and G. Toscani. Kinetic models for goods exchange in a multi-agent market. *Physica A: Statistical Mechanics and its Applications*, 499:362–375, 2018.

[2] A. Chatterjee, B. K. Chakrabarti, and R. B. Stinchcombe. Master equation for a kinetic model of a trading market and its analytic solution. *Physical Review E*, 72(2):026126, 2005.

[3] S. Cordier, L. Pareschi, and C. Piatecki. Mesoscopic modelling of financial markets. *Journal of Statistical Physics*, 134(1):161–184, 2009.

[4] S. Cordier, L. Pareschi, and G. Toscani. On a kinetic model for a simple market economy. *Journal of Statistical Physics*, 120(1-2):253–277, 2005.

[5] J. H. Ferziger and H.G. Kaper. *Mathematical Theory of Transport Processes in Gases*. North-Holland, 1972.

[6] K. Kanazawa, T. Sueshige, H. Takayasu, and M. Takayasu. Derivation of the Boltzmann equation for financial Brownian motion: Direct observation of the collective motion of high-frequency traders. *Physical review letters*, 120(13):138301, 2018.

Description

Motivation

Kinetic theory is a versatile formalism for systems with many degrees of freedom (DoFs). It stems from the original statistical mechanics devised by Boltzmann, Maxwell, and Gibbs and holds its name from the (incredible) achievement of relating ensemble properties -namely thermodynamics (density, temperature, pressure) and transport properties (viscosity, thermal conduction)- to velocities of molecules, i.e. microscopic kinetic energy. To describe a system (gas, mist, traffic, market) with many interacting particles (molecules, drops, cars, traders) a distribution f of the DoFs of a single particle is used. The "collision" term describes the evolution of f at a given state due to the effect of (all the particles in) all other states. In most cases the collision operator holds the modeling and resolution complexity.

The Boltzmann collision operator is a beautiful piece of physics and a canonical exercise of mathematical modeling, which brings irreversibility & entropy in a very subtle (and debated) way. It lends itself to a wealth of stimulating model extensions.
Linking many “irrelevant” microscopic phenomena to the relevant macroscopic DoFs is desirable to build and brace economic and social sciences, e.g. to bridge the gap between micro- and macro-economics. In the spirit of econophysics, some recent works resort to kinetic theory in order to model markets:

- [2, 4] with 1 tradable asset (wealth) where random profits arise from each trade,
- [3] with 2 assets (stock/bond) and a supply-demand price model,
- [1] with multiple agent types, namely dealer/speculator,
- [6] focusing on price formation in high-frequency trading.

The first three approaches are based on a wealth variable, hence following a conservation law. In the latter, the price variable is a positional rather than a conserved quantity (wealth): the authors speak in terms of bid-ask distance, which drives the transaction (collision) probability. In all these cases, there is a collision operator driving the dynamics of the distribution. An equilibrium distribution can be explicitly provided, e.g. the well-known Pareto distribution for wealth in [2, 4].
Finally let us highlight that kinetic equations naturally lend themselves to Monte-Carlo simulation. This method was actually created by J. von Neuman and co-workers to solve kinetic equations for neutrons. In most cases however, the collision operator precludes a parallel resolution of paths. Kinetic problems can alternatively be tackled by taking some moments of the distribution, closing the dependencies in higher order terms and solving for the resulting system of PDEs. The equations of fluid dynamics (compressible Euler, Navier-Stokes) can be so derived, with the three first moments in molecular velocity being the local fluid's mass, momentum, & internal energy and the closures for higher order moments (pressure tensor, heat flux) resorting to an equation of state [5].

Technical Details (part 1)

We describe a gas with many molecules at velocity $\mathbf{c}$ using $f(t, \mathbf{x}; \mathbf{c})$, which depends on space-time $(t, \mathbf{x})$. Normalisation is so that $\int_{\mathbf{c}} f(t, \mathbf{x}; \mathbf{c}) = n(t, \mathbf{x})$ is the number of molecules at $(t, \mathbf{x})$. The rate of change $\mathrm{d}f/\mathrm{d}t$ (expanded using the chain rule) is caused by collisions:

$$\partial_t f + \mathbf{c} \cdot \partial_{\mathbf{x}} f = \mathcal{C}_{\mathrm{B}}(f, f)$$

where $\mathrm{d}_t\mathbf{x}$ has been replaced by $\mathbf{c}$ and $\mathrm{d}_t\mathbf{c} \cdot \partial_{\mathbf{c}} f = 0$ considering no body forces. $\mathcal{C}_{\mathrm{B}}$ is the Boltzmann collision operator. To derive the latter, we assume collisions as instantaneous transformations: $f(\mathbf{c}) \triangleq f$ and $f(\mathbf{c}_\star) \triangleq f_\star$ disappear when colliding together, immediately replaced by $f(\mathbf{c}') \triangleq f'$ and $f(\mathbf{c}'_\star) \triangleq f'_\star$. The corresponding 4 velocities are tied by the dynamics of (hard) sphere collisions: number, momentum and kinetic energy are conserved. The rate of change of f can be obtained by multiplying $(f'f'_\star - ff_\star)$ by a frequency[1] σV and integrating over DoF $\mathbf{c}_\star \in \mathbb{R}$ while $\mathbf{c}'$ and $\mathbf{c}'_\star$ depend on $(\mathbf{c}, \mathbf{c}_\star)$:

$$\mathcal{C}_{\mathrm{B}}(t, \mathbf{x}; \mathbf{c}) = \int_{\mathbf{c}_\star} \sigma V \left(f'f'_\star - ff_\star \right)$$

[1] Here, collisions are driven by the norm of the velocity difference between the two initial partners $V = |\mathbf{c}_\star - \mathbf{c}|$ and a cross-section σ. A dimensional analysis of collision frequency ν gives the scaling $\nu \sim \sigma V f$.

As a result $\int_{\mathbf{c}}(1,\mathbf{c},\frac{1}{2}\mathbf{c}^2)\mathcal{C}_{\mathrm{B}}(t,\mathbf{x};\mathbf{c}) = (0,0,0)$ so that macroscopic number, momentum and total energy $\int_{\mathbf{c}}(1,\mathbf{c},\frac{1}{2}\mathbf{c}^2)f(t,\mathbf{x};\mathbf{c}) \triangleq (n,n\mathbf{u},E)(t,\mathbf{x})$ are conserved in the collision process.

Questions

- **Q1:** ($\mathcal{H}$-theorem) *Prove that $\mathcal{H} = f\ln f$ decreases over time. (Hint: After exploiting the symmetries, consider the sign of $(x'-x)\ln(x/x')$.)*
- **Q2:** (equilibrium) *Can you find a distribution that minimizes $\mathcal{H}$? (Hint: Use a linear combination of microscopic collision invariants.)*

The $\mathcal{H}$-theorem was discovered by Boltzmann and he believed $\mathcal{H}$ was equal to (the opposite of) Clausius' entropy. The emergence of irreversibility is puzzling and is actually hidden in the molecular chaos (Stosszahlansatz) assumption of considering uncorrelated molecules after collision $f(\mathbf{c}')f(\mathbf{c}'_\star)$ rather than a two-point distribution $f^{(2)}(\mathbf{c}',\mathbf{c}'_\star)$. Most importantly, the equilibrium distribution is the celebrated Maxwell-Boltzmann distribution, a state near which gases spend most of their time (except in high vacuum or in space).

Technical Details (part 2)

Now consider a market as in [3] as our new system, with many buyers and sellers. The only phase space variable is wealth w, following a "collision" dynamics when a trade happens:

$$\begin{aligned} w' &= (1-\gamma)w + \gamma w_\star + \eta w \\ w'_\star &= (1-\gamma)w_\star + \gamma w + \eta_\star w_\star \end{aligned}$$

with the usual pre-/post-trade notations and γ the transaction propensity coefficient a constant between 0 and 1. η and $\eta_\star$ are random variables accounting for the benefits of transactions, following the same distribution μ (e.g. normal with variance σ^2 and zero mean). The trade takes place only if w', $w'_\star \geq 0$ to avoid debt. The kinetic equation finally reads:

$$\partial_t f(t;w) = \int_{w_\star}\int_{\eta}\int_{\eta_\star} \mu(\eta)\mu(\eta_\star)\mathbb{1}(w)\mathbb{1}(w_\star)\big(f(w')f(w'_\star) - f(w)f(w_\star)\big)$$

Questions

- **Q3:** *What could the existence of an entropy theorem be good for in such a non-thermodynamics context?*
- **Q4:** *Consider* $\langle w+w_{\star}\rangle$ *and write the evolution equation for the system's total wealth.*
- **Q5:** *Prove that, under suitable assumptions, the tail of* f *is a Pareto distribution* $f(w) \sim \frac{1}{w^{1+\alpha}}$ *with* $\alpha > 1$.