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Mastering Complexity

Mathematical Challenge June 2019

Interest rate modelling

References

[1] Alexandre Antonov, Michael Konikov, and Michael Spector. The free boundary sabr: Natural extension to negative rates. *SSRN Electronic Journal*, 01 2015.

[2] Zorana Grbac and Wolfgang J Runggaldier. *Interest rate modeling: post-crisis challenges and approaches*. Springer, 2015.

[3] Patrick Hagan, Deep Kumar, Andrew Lesniewski, and Diana E. Woodward. Managing smile risk. *Wilmott Magazine*, 1:84–108, 01 2002.

Description

Motivation

The credit crisis in 2007-2008 and the Eurozone sovereign debt crisis in 2009-2012 have massively impacted all financial markets, causing a reconsideration of how their theoretical models should be developed [2]. The key features that were put forward by the crises are *counterparty risk*, i.e. the risk of a counterparty failing to fulfill its obligations in a financial contract, and *liquidity* or *funding risk*, the risk of excessive costs of funding a position in a financial contract due to the lack of liquidity. These issues have particularly affected the fixed-income market.
Moreover, many central banks have set interest rates to below zero, incentivising banks to lend money more freely, businesses and individuals to invest money rather than pay a fee to keep it safe. Black model, widely used in the pre-crisis period was thus rendered unusable due to its assumption of log-normally distributed rates.
In this challenge, we will cover some recent work on interest rate models tackling the above mentioned issues and its application to derivative pricing.

Negative interest rates

Let us consider some standard approaches for univariate forward interest rate modelling:

- CEV (constant elasticity of variance) model

$$dF_t = \sigma F_t^{\beta} dz_t$$

  - dz_t is a Wiener process, σ denotes the volatility, $\beta \in [0, 2]$
  - Special case: Black's model
    * $\beta = 1$, forward rate lognormally distributed, strictly positive values
    * σ - st. dev. of the relative interest rate change
  - Special case: Bachelier model
    * $\beta = 0$, forward rate normally distributed, allows for negative rates
    * σ - st. dev. of the absolute interest rate change

- SABR model [3]

$$\begin{aligned} dF_t &= \sigma_t F_t^{\beta} dz_t \\ d\sigma_t &= \nu \sigma_t d\omega_t \\ d\omega_t dz_t &= \rho dt \end{aligned} \tag{1}$$

  - $dz_t, d\omega_t$ are Wiener processes, ρ is the correlation coefficient, σ is the volatility, ν the volatility of the volatility, $\beta \in [0, 1)$.
  - extension of CEV model to allow for stochastic volatility.

Bachelier model is the only one allowing for negative rates. However, the distribution is inappropriately skewed towards negative values, since large negative rates are highly unlikely. Another way to account for close-to-zero or negative values is to introduce a shift s_t to the forward rate (e.g the shifted Black model reads $d(F_t + s_t) = \sigma\,(F_t + s_t)\,dz_t$), effectively setting the lowest possible rate to $-s_t$ rather than zero.
In [1], they suggest a free boundary SABR model with following modifications to (1):

$$dF_t = \sigma_t \left|F_t\right|^{\beta} dz_t, \quad \beta \in [0, 0.5) \tag{2}$$

- **Q1:** *Compare the approach in [1] with adding a deterministic shift to F_t.*
- **Q2:** *What are the caveats of single-curve interest rate modelling post-crisis?*
- **Q3:** *Describe pricing of a European interest rate call option (rate follows shifted SABR), assuming knowledge of the standard pricing formula.*

Arbitrage violations

Financial crisis has also caused the appearance of previously unobserved anomalies. Interest rates whose dynamics were very closely following each other have started diverging substantially, prompting the introduction of various spreads:

- Libor-OIS spread (same maturity)
- Libor-OIS swap spread
- basis swap spreads (e.g Libor rates of different maturities)

Growing spreads potentially present arbitrage opportunities in the market, which has served as motivation for development of multi-curve models. We will explore the Heath-Jarrow-Morton (HJM) framework presented in [2]. Instead of directly modelling liquidity or counterparty (i.e modelling why the yield curves differ), the approach considers different dynamics for forward rates of different tenors.

In order to develop an arbitrage-free model in the multiple curve HJM setup, we first consider a model for the OIS prices, used as a discounting curve (see section 1.3.1 [2] for more details). Let $f(t,T)$ denote the instantaneous, continuously compounded forward OIS interest rate and $g(t,T)$ the forward rate spread at time t with maturity $T \geq t$, $T \in [0,\overline{T}]$, $\overline{T}$ is a finite time horizon. Continuously compounded LIBOR forward rates can be expressed as $\overline{f}(t,T) = f(t,T) + g(t,T)$. Corresponding fictitious bond prices can then be derived as $\overline{p}(t,T) = exp[-\int_t^T \overline{f}(t,u)du]$. Under martingale modeling assumption on probability space $(\Omega, F, (F_t)_{0 \leq t \leq -T}, Q)$, the forward rates and spreads satisfy (analogous for g and $\overline{f}$):

$$df(t,T) = \mu(t,T)dt + \sigma(t,T)d\omega_t$$

where $d\omega_t$ is a Wiener process. Under required regularity assumptions, we get the following dynamics for OIS bond prices:

$$dp(t,T) = p(t,T)\left[\left(r_t - A(t,T) + \frac{1}{2}\left|\Sigma(t,T)\right|^2\right)dt - \Sigma(t,T)d\omega_t\right] \tag{3}$$

with $A(t,T) = \int_t^T \mu(t,u)du$, $\Sigma(t,T) = \int_t^T \sigma(t,u)du$, $r_t = f(t,t)$. Discounted OIS bond prices have to be martingales under Q, which leads to the condition $A(t,T) = \frac{1}{2}|\Sigma(t,T)|^2$. We can derive analogous expressions for $\overline{p}(t,T)$ and $g(t,T)$, but not directly apply the Martingale condition to $\overline{p}(t,T)$ [2].

LIBOR forward rates can be expressed as:

$$L(t;T,T+\Delta) = \frac{1}{\Delta}E^{T+\Delta}\left[\left(\frac{1}{p(T,T+\Delta)} - 1)\right| F_t\right] = \frac{1}{\Delta}\left(\frac{p^{\Delta}(t,T)}{p^{\Delta}(t,T+\Delta)} - 1\right)$$

where $\overline{p}(t,T)$ is replaced with $p^{\Delta}(t,T)$ to emphasize that the relationship between forward rates and the bonds is preserved at all levels t. $L(t;T,T+\Delta)$ is a $Q^{T+\Delta}$-martingale, implying that the process $\nu_{t,T}^{\Delta} := \frac{p_{t,T}^{\Delta}}{p_{t,T+\Delta}^{\Delta}}$ is also a $Q^{T+\Delta}$-martingale. This martingale condition is directly implied by the absence of arbitrage imposed on the FRA contracts [2]. As usual, the no-arbitrage condition can be derived by determining the drift in the dynamics of $\nu_{t,T}^{\Delta}$ under $Q^{T+\Delta}$ and setting it equal to zero:

$$A^{\Delta}(t,T+\Delta)-A^{\Delta}(t,T) = -\frac{1}{2}\left|\Sigma^{\Delta}(t,T+\Delta)-\Sigma^{\Delta}(t,T)\right|^{2}+\left\langle\Sigma(t,T+\Delta)\Sigma^{\Delta}(t,T+\Delta)-\Sigma^{\Delta}(t,T)\right\rangle \tag{4}$$

Proof can be found in section 3.2 of [2].

- **Q4:**_Obtain the expression for the induced short-rate spread assuming Vasicek-type volatility. (see section 3.2.2.2 of [2])._
- **Q5:**_Derive the expression for Libor-indexed payer swap option using the above results._
